Constraint Propagation Problems

Problem 1

Consider the following constraint network, where all variables have domain {1,2,3}, except variables A and I, whose domain is {0,1,2,3}. The binary constraints labelled with e (resp. o) are satisfied if one of the variables take value 0 or the sum of the variables is even (resp. odd).

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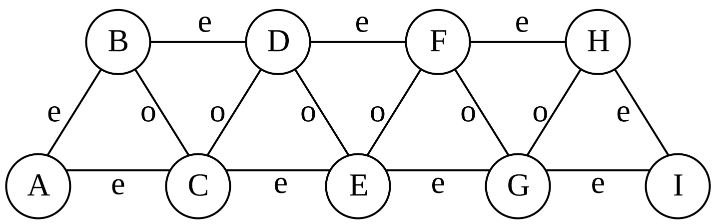


a) What would be the pruning of the variables domains if node-consistency is maintained? And arc-consistency? Justify.

b) Show that path-consistency would be able to fix the value of some variables? Which ones?

c) Justify whether maintaining arc-consistency would be sufficient to obtain solutions of the problem without backtracking. And path consistency?

Proposed Solution:

a) Maintaining node-consistency only affects unary constraints. As there are none, no pruning is achieved.

For every variable (node) all and each of its neighbours have values with the same and different parity. Hence every value of this variable has some support in all its neighbours, so again no pruning is achieved.

b) Considering path consistency, take the pair of variables <A, B>. Because C has the same parity of A and different parity from B, this means that A and B should have different parities. But given the binary constraint between A and B, these variables either have values with the same parity, or A = 0, hence it must be A = 0.

A similar reasoning would fix the value of variable I to I = 0.

c) The ordering A, B, C, … I induces a width of 2 in the graph (every node has at most 2 neighbours with lower ranking. Hence if variables are labelled in this order, no backtracking occurs. As seen above, A is fixed to 0. Then, given constraint A-B, B might take any value v. Then propagation of the other constraints will enforce the other variables to keep only values with compatible parities, i.e. D, F and H will keep values with the same parity of v and C, e and G will keep values with parity different from v, whereas I is already fixed to 0.

Considering arc-consistency, that has not fixed A and I to 0, labelling A with a value different from 0 will cause backtracking since propagation of constraints A-B and A-C will enforce the values remaining in B and C to have the same parity of the chosen value for A, but this is contradictory with the constraint B-C.

Problem 2

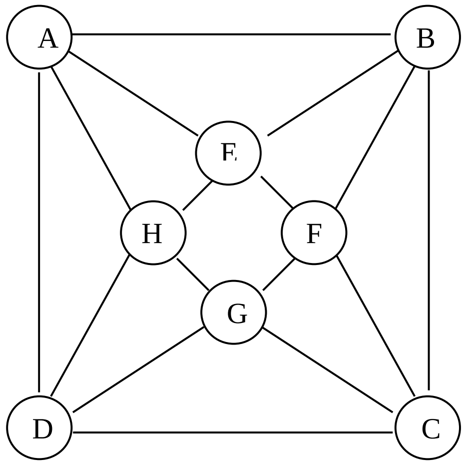
Consider the following constraint networks where variables A, B, C and D have domain {1,2,3} and variables E, F, G and H have domain {2,3,4} The binary constraints shown are all difference constraints ($\neq$).

a) Show that the constraint network is node- and arc-consistent. Justify.

b) Assume arc-consistency is maintained on the constraint network. What would be the result of propagation once A is set to 2?

c) Assume now that variables A and B are restricted to the domain {2,3} (and the others keep their previous domains. Show that the problem becomes impossible.

d) Do you think this impossibility would be obtained, without backtracking, by arc-consistency? And path consistency? Justify.



Proposed Solution:

a) Maintaining node-consistency only affects unary constraints. As there are none, no pruning is achieved.

   All variables have at least two values in their domain. Hence, every value of a variable has at least one support in any of its neighbouring variables, so no pruning is achieved by maintaining arc-consistency.

b) Once A is set to 2, maintaining node-consistency, would prune the domains of variables B and D to {1,3} and the domains of variables H and E to {3,4}. Since all variables, except A, would have at least two values in their domain, no further pruning would be achieved.

c) If variables A and B have domain {2,3}, and they are different, variable E should be fixed to 4, since this is the only value in its domain which is different from 2 and 3. Hence variables H and F will have their domain pruned to {2,3}. Hence variables A and H have domain {2,3} which fixes the value for D to D = 1. Similarly, since B and F have domain {2,3} the value for C is fixed to C = 1. But C and D must be different so the problem is impossible.

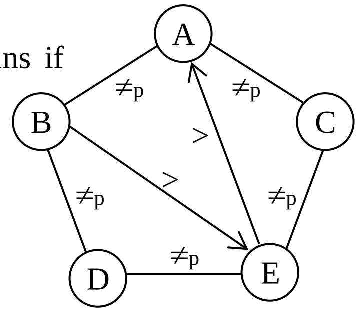
d) Propagation of constraints of difference to maintain arc-consistency only prunes domains if the domain of one variable is fixed. Since no variables are initially fixed, no pruning is achieved and the network is arc-consistent (but not satisfiable).

   Consider pair A-E with domains A in {2,3} and E in {2,3,4}. Any partial label with this pair, can only be extended to variable B, with domain {2,3} if variable E takes value E = 4, thus fixing its value. Hence variables H and F have their domains pruned to {2,3}. Now, any partial label concerning variables H and A could only be extended to D if D is fixed to D = 1. Similarly, any partial label regarding variables F and B could only be extended to C if it is fixed to D = 1. Fixing the values of C and D to 1 contradicts the constraint C $\neq$ D, and hence the impossibility of the problem is detected by maintaining path-consistency.

Problem 3

Consider the following constraint network, where all variables have domain {1,2,3,4}. The binary constraints labelled with ≠p are satisfied if the constrained variables have diffeernt parity (i.e. one is even and the other odd). The binary constraints represented by directed arcs A → B should be read as X > Y. For the following questions provide the answers and the adequate justifications.

a) Is the constraint network satisfiable?
b) What would be the pruning of the variable domains if bounds-consistency is maintained?
c) And arc-consistency?
d) And path-consistency?



Proposed Solution:

a) Checking the parity constraints on the variables we notice that A should have different parity of B and C. Hence, both D and E should have the same parity of variable A. But this is impossible, as D and E should have different parity. Hence, the problem is impossible.

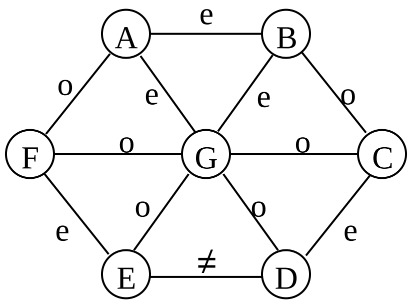
b) Maintaining arc-consistency on the inequality constraints (B>E and E>A) prunes the domais of variables A, E and B to respectively {1,2}, {2,3} and {3,4}. Although reduced, these domains keep values of both even and odd parities. Hence the upper bound of all variables (be it, 2, 3 or 4) is supported by some value of each and all of the neighbouring variables. And so does the lower bound (be it, 1, 2 or 3), leading to no reduction of the domains of variables C and D.

c) Maintenance of arc-consistency leads to the same pruning of bounds-consistency for this problem. The reasoning above regarding the upper- and lower-bound values could be extended for values 2 and 3 of variables C and D (that are not bounds of the variables) but have support in each of the neighbouring variables.

d) Considering labels of the variable pair [A , D], we notice that to be consistently extended to variable B, they must have the same parity. Hence there is an implicit constraint $A =_p D$ that is elicited by maintaining path consistency. Similarly, another implicit constraint, $A =_p E$, is elicited. Hence, any consistent label regarding variables D and E, that must have values with different parities, may be extended to variable A. Hence, maintaining path-consistency is sufficient to prove that the problem is impossible.

Problem 4

Consider the following constraint network, where nodes correspond to variables and edges to binary constraints. Edges labelled with **e** and **o** denote, respectively, constraints imposing that the sum of the variables is even and odd. Edges labelled with a ≠ correspond to the usual difference constraint. All variables have domain {1,2,3}.



a) What is the domain pruning achieved when node-, arc- and path-consistency is maintained? Justify.

b) Consider now that the constraint between variables A and B becomes of type "o". Verify that the problem becomes unsatisfiable and show what type of consistency should be maintained to detect such unsatisfiability without labelling the variables.

c) For a generic constraint network with the above topology, where enumeration was needed, what variable ordering would you use to do such enumeration? How many variables should be labelled, to reach a backtracking free labelling of the remaining network? Justify.

Proposed Solution:

a) Maintaining node-consistency only affects unary constraints. As there are none, no pruning is achieved.

All variables have at least two values in their domain and with different parity. Hence, every value of a variable has at least one value that supports it in any of its neighbouring variables, regardless of the binary constraint (e, o or ≠). Hence no pruning is achieved by maintaining arc-consistency.

To be extended consistently to variable G, any label for the pair of variables D and E should enforce values with the same parity for these variables (D and E). But since they must be different, the only possible pairs for the variables are <D=1, E=3> or <D=3, E=1>, and value 2 is thus pruned from the domain of both variables, which also fixes G to value 2. Further propagation will prune the domains of C and F to {1,3}, and the domains of A and B to the singleton {2}.

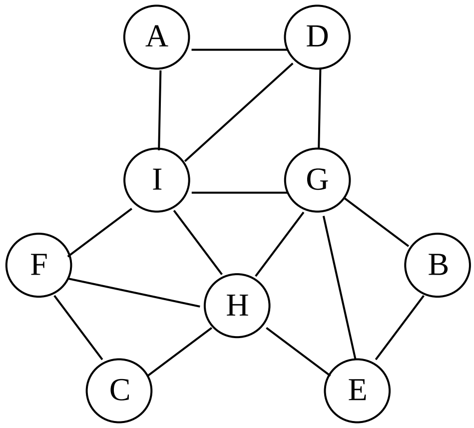
b) For the reason explained above, maintenance of arc-consistency would not prune any domain, since all domains remain with at least two values of different parity.

Now, maintaining of path-consistent would guarantee that any consistent labelling of the variable pair [A,B] could be extended to variable G. But any consistent label of variables A and B enforces A and B to have values with different parity (e.g. A=1 and B = 2). But then no value for G has different parity from the values chosen for A and B, thus pruning all values from the domains of A and B, thus detecting the unsatisfiability of the problem.

c) In general, a good static variable selection heuristics should adopt some ordering that induces minimal with to the network graph. In this case, this recommends starting with variable G, so that all other variables have at most two neighbouring variables with lower rank. Once G and another variable are labelled, the remaining graph becomes a tree (in fact a sequence) and any labelling becomes backtracking free.

Problem 5

Consider the following constraint network, where all variables have domain {1,2,3} and the binary constraints are constraints of difference (≠).



a) Show that no type of consistency (node-, arc- or path-) may prune the domains of the variables. Suggestion: analyse the solutions of the problem.

b) Assume that variables are labelled according to the ordering [A, B, C, D, E, F, G, H, I]. Show that maintenance of arc-consistency does not guarantee a backtracking free labelling. And path-consistency? Justify your answer.

c) Present an ordering of the variables, static or dynamic, that guarantees a backtracking free search in conjunction with arc consistency. Justify your answer.

d) Could you infer extra-constraints of equality (=) for the problem (keeping equivalence)? Should you make these constraints explicit? Why?

Proposed Solution:

a) Because A and G are variables connected to the same two variables (I and D) that must be different and there are only 3 values in the domains of the variables, there is an implicit constraint that enforces A = G in any solution. Similarly we can btain the following sets of equal variables: A = F = G, B = H = D, and C = E = I. A can be labelled with an arbitrary value in its domain, so no pruning can be achieved by any type of propagation. Similarly, B and C can take any value provided they are different between them and wrt A, so there are solutions for all values of the variables and hence no value can be pruned from the domain of the variables.

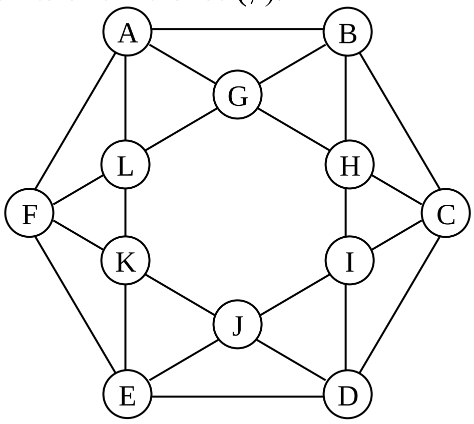
b) From the above A, B and C should all take different values. But if we label A = 1 and B = 1 and C = 1, constraint propagation would prune the domains of all the other variables to {2,3}, and since there are two values in their domains no more pruning would be achieved through propagation of ≠ constraints. Only when variable D is labelled to 2 (or 3) would an inconsistency by detected: on one hand, it should be I = G = 3 (or 2) since they must both be different from D; on the other hand, the constraint I ≠ G prevents variables I and G from taking the same value.

c) Starting the enumeration with the variables with more neighbours (higher degree) is usually a good variable selection heuristics. In this case this suggests starting the labelling with variables G, H and I. As seen before, propagation after labelling G to any of the values in its domain, narrows the domains of H and I to the remaining 2 values. Propagation after labelling H with one of the values, leads to fixing I with the remaining value. From then on, all the values of the other variables will be fixed as discussed above (A = F = G, B = H = D, and C = E = I). Hence a good ordering would start by G, H, I (in any order) followed by the other variables (again in any order).

d) There are implicit ≠ constraints as seen above: A = F = G, B = H = D, and C = E = I. Making them explicit would guarantee any labelling order to be backtracking free: for example, labelling a variable in one of the groups to some value (say A = 1) would fix the values of the other variables of the group (i.e. F = G = 1) to 1, thus narrowing the domain of all the other variables to{2,3}. Then labelling a variable of one of the remaining groups to some value in this narrowed domain (say B = 2) fixes the values of the remaining variables of the group (H = D = 2) and also the variables of the other group (C = E = I = 3).

Problem 6

Consider the following constraint network where all the "external" variables (A a F) have domain {1,2,3} and the others domain {1,2}. All constraints are constraints of difference (≠).



a) Show that the constraint network is not satisfiable.

b) Show that mere maintenance of node- and arc-consistency would not detect such unsatisfiability.

c) And path-consistency?

d) Show that the problem becomes satisfiable if all variables have initial domain {1,2,3}.

e) Does any type of consistency achieve domain reduction for any of the variables?

f) Explain what type of consistency and variable heuristics seem more adequate to this network. Will the combination avoid backtracking during search?

g) How many solutions does the problem have? What additional constraints could be imposed to avoid symmetries? Justify.

Proposed Solution:

a) Because L and G must be different and there are only 2 values in their domains, one must take value 1 and the other value 2. Hence, variable A, that must be different from both, must take value 3. Similarly, variables G and H, force variable B to take value 3. But this is not possible since variables A and B must be different.

b) There are no unary constraints, so imposing node-consistency does not prune any value from the variables' domain. Imposing arc-consistency on constraints of difference does not prune any value of the variables' domains, as long as there at least two values in the domain of each variable, as is the case here.

c) Maintaining path consistency would detect unsatisfiability. In fact, the only labels on variables A and L (ie. <A-v_a, L-v_l>) that can be extended to variable G, guarantee that v_a =3. Similarly, the only labels <B-v_b, H-v_h> that can be extended to variable G, guarantee that v_a = 3, contrary to constraint A ≠ B.

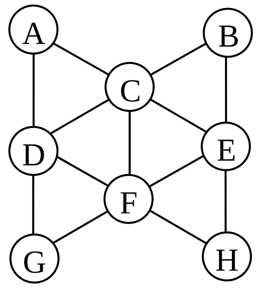
d) A possible solution would start assigning two different values to variables A and B, which would fix all the other variables. For example, A = 1 and B = 2, enforce G = 3, L = 2, F = 3, K = 1, E = 2, J = 3, D = 1, I = 2, C = 3, and H = 1.

e) Through symmetry, it is easy to see that assigning different pairs of values to A and B would impose all possible values to variable G (for example A=1, B=3, G=2 or A=2, B = 3, G = 1). Hence there are no values in the domain of the variables that may be pruned by any type of consistency.

f) Given the symmetry of the problem each combination of different values for A and B would lead to a single solution. There are 6 combinations for distinct values of A and B, so there are 6 distinct solutions for the problem.

Problem 7

Consider the constraint network below, where nodes correspond to variables and arcs to binary constraints.



a) Assuming that all constraints are constraints of difference (≠), show that the network has no solutions if all variables have domain {1,2}. What type of consistency (node-, arc-, path- or higher k- consistency) would allow this conclusion to be inferred, without enumeration of the variables?

b) Now assume that variables C and F have domain {1,3} and all other variables have domain {1,2,3}.

i. Show that maintaining arc-consistency does not prune the domain of any of the variables.

ii. In contrast, show that maintaining path-consistency grounds (fixes the value of) some variables. Which variables and what are their values? Are there any other domains pruning?

iii. Show that the (non-binary) constraint A+B+G+H = 9 is not satisfiable. Would path-consistency on the binary constraints be sufficient to infer this unsatifiability)

c) Assume now that i. all binary constraints in the network are arbitrary, ii. that all variables have *d* values in their domain, and iii. arc-consistency is maintained.

i. Present a static ordering of the variables that should lead to an efficient resolution of the problem. Justify your answer.

ii. Assuming the same ordering for variable labelling, can you present a weaker type of consistency that, if maintained, would lead to no more backtracks during the labelling of the variables? Justify.

Proposed Solution:

a) Variables A, C and D must all be difefernt from each other, which is impossible if their domain is {1,2}. Path-consistency would detect this, as no label <A.v-a, C-v_c> could be extended to variable D.

b)

i. As long as the domains of two variables that must be different have at least 2 values, arc-consistency will not achieve any pruning.

ii. If variables C and F take values 1 and 3, then variables D and E must take value 2. Path-consistency would detect this fixing of the variables. Moreover, A, B, G and H would have value 2 removed from their domain.

iii. From the previous item, variables A, B, G and H have only even values in their domain, so their sum should be even, so the constraint A+B+G+H = 9 is not satisfiable. Although easy to detect symbolically, this unsatisfiablity, it should only be possible to detect by 4-consistency, as no combination of values of 3 of the variables could be extended to the fourth variable.

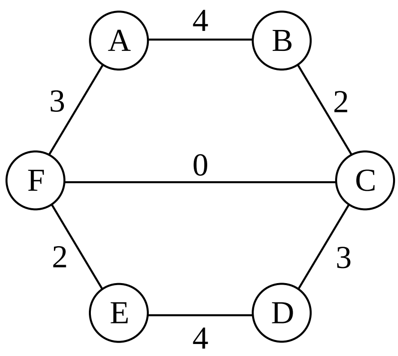
c)

i. Start labelling one of the variables C or F, then the other. The remaining variables form two independent linear graphs whose satisfiability is guaranteed by arc-consistency.

ii. Directed arc consistency is sufficient. In fact it is sufficient that every value of C and F have support on the other variables, since C and F are the first to be labelled.

Problem 8

Consider the following constraint network, where all variables have domain {1,2,3,4,5,6}. The binary constraints, labelled with an integer k, are satisfied if this is the absolute distance between the constrained variables (i.e. if k = 2 and one variable is 3, the other must have values 1 or 5. In particular equality is obtained with k=0). For the following questions provide the answers and the adequate justifications.



a) What would be the pruning of the variable domains if node-consistency is maintained?

b) And bounds-consistency?

c) And arc-consistency?

Proposed Solution:

a) No pruning is achieved as there are no unary constraints.

b) No pruning is achieved either. All bounds of the domain of the variables have a support on any variable they are constrained with. For example, A=0 is supported by B = 4 and F = 3, and A = 6 is supported by B = 2 and F = 3.

c) Arc consistency would prune value 3 of variables A and B. Similarly, value 3 is also pruned for variables D and E.