Intervals, Interval Arithmetic and Interval Functions

Jorge Cruz

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Basic Concepts

Constraint. A constraint c is a pair (s,ρ), where s is a tuple of m variables $<x_1, x_2,\ldots, x_m>$, the constraint scope, and ρ is a relation of arity m, the constraint relation. The relation ρ is a subset of the set of all m-tuples of elements from the Cartesian product $D_1 \times D_2 \times \ldots \times D_m$ where D_i is the domain of the variable x_i:

$$\rho \subseteq \{<d_1, d_2, \ldots, d_m> \mid d_1 \in D_1, d_2 \in D_2, \ldots, d_m \in D_m\}$$ ❑

Constraint Satisfaction Problem. A CSP is a triple $P=(X,D,C)$ where X is a tuple of n variables $<x_1, x_2, \ldots, x_n>$, D is the Cartesian product of the respective domains $D_1 \times D_2 \times \ldots \times D_n$, i.e. each variable x_i ranges over the domain D_i, and C is a finite set of constraints where the elements of the scope of each constraint are all elements of X. ❑

Basic Concepts

Constraint Satisfaction. Let $P=(X,D,C)$ be a CSP. Let (s,ρ) be a constraint from C and d an element of D:

d satisfies (s,ρ) iff $d[s] \in \rho$ ❑

Solution. A solution to the CSP $P=(X,D,C)$ is a tuple $d \in D$ that satisfies each constraint $c \in C$, that is:

d is a solution of P iff $\forall_{c \in C}$ d satisfies c ❑

Consistency. A CSP $P=(X,D,C)$ is consistent iff it has at least one solution (otherwise it is inconsistent):

P is consistent iff $\exists_{d \in D}$ d is a solution of P ❑

Basic Concepts

Continuous Constraint Satisfaction Problem. A CCSP is a CSP P=(X,D,C) where each domain is an interval of $\mathbb{R}$ and each constraint relation is defined as a numerical equality or inequality:

i) D=<D_1,…,D_n> where D_i is a real interval $(1 \leq i \leq n)$

ii) $\forall_{c \in C}$ c is defined as $e_c \diamond 0$ where e_c is a real expression and $\diamond \in \{\leq,=,\geq\}$ ❑

***R*-interval.** A real interval is a connected set of reals. Let $a \leq b$ be reals, the following notations for representing real intervals will be used:

$[a..b] \equiv \{r \in \mathbb{R} \mid a \leq r \leq b\}$ $\quad$ $(a..b) \equiv \{r \in \mathbb{R} \mid a < r < b\}$

$(a..b] \equiv \{r \in \mathbb{R} \mid a < r \leq b\}$ $\quad$ $[a..b) \equiv \{r \in \mathbb{R} \mid a \leq r < b\}$

$[a..+\infty) \equiv \{r \in \mathbb{R} \mid a \leq r\}$ $\quad$ $(a..+\infty) \equiv \{r \in \mathbb{R} \mid a < r\}$

$(-\infty..b] \equiv \{r \in \mathbb{R} \mid r \leq b\}$ $\quad$ $(-\infty..b) \equiv \{r \in \mathbb{R} \mid r < b\}$

$(-\infty..+\infty) \equiv \mathbb{R}$ $\quad$ $\varnothing \equiv \{\}$

The notation <a..b> will represent a nonempty real interval of any of the defined forms. ❑

Representation of Continuous Domains

F-Numbers, Intervals and Boxes

***F*-numbers.** Let $\boldsymbol{F}$ be a subset of $\mathbb{R}$ containing the real number 0 as well as finitely many other reals, and two elements (not reals) denoted by $-\infty$ and $+\infty$:

$$\boldsymbol{F} = \{r_0,\ldots,r_n\} \cup \{-\infty,+\infty\} \qquad \text{with } 0 \in \{r_0,\ldots,r_n\} \subset \mathbb{R}$$

The elements of $\boldsymbol{F}$ are called F-numbers. ❑

F is totally ordered:

any two real elements of F are ordered as in $\mathbb{R}$

$-\infty < r < +\infty$ for all real element r

If f is an F-number, f^- and f^+ are the two F-numbers immediately below and above f in the total order:

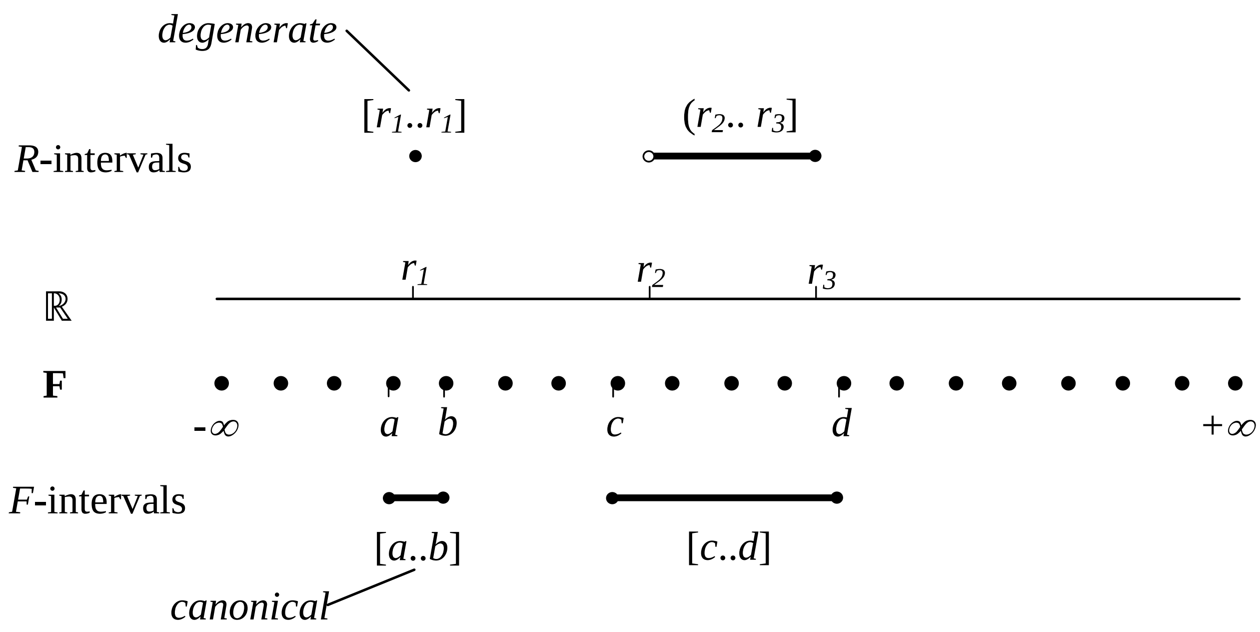
$-\infty^- = -\infty$ and $+\infty^+ = +\infty$

$-\infty^+$ is the smallest real in $\boldsymbol{F}$ and $+\infty^-$ is the largest real in $\boldsymbol{F}$

Representation of Continuous Domains

F-Numbers, Intervals and Boxes

***F*-interval.** An *F*-interval is a real interval <*a*..*b*> where *a* and *b* are *F*-numbers.
In particular, if $b=a$ or $b=a^+$ then <*a*..*b*> is a ***canonical F*-interval**. ❑



In the following we only consider closed *F*-intervals: [*a*,*b*]
If *a*=*b* the interval is degenerated and is represented as *a*

Representation of Continuous Domains

F-Numbers, Intervals and Boxes

Extending the interval concepts to multiple dimensions:

***R*-box.** An *R*-box *BR* with arity *n* is the Cartesian product of *n* *R*-intervals and is denoted by $<IR_1,\ldots,IR_n>$ where each IR_i is an *R*-interval:

$$BR = \{<r_1, r_2, \ldots, r_m> \mid r_1 \in IR_1, r_2 \in IR_2, \ldots, r_n \in IR_n\}$$ ❑

***F*-box.** An *F*-box *BF* with arity *n* is the Cartesian product of *n* *F*-intervals and is denoted by $<IF_1,\ldots,IF_n>$ where each IF_i is an *F*-interval:

$$BF = \{<r_1, r_2, \ldots, r_m> \mid r_1 \in IF_1, r_2 \in IF_2, \ldots, r_n \in IF_n\}$$

In particular, if all the *F*-intervals IF_i are canonical then *BF* is a ***canonical F*-box**. ❑

Representation of Continuous Domains

Interval Operations and Basic Functions

All the usual set operations may also be applied on intervals:

$\cap$ (intersection)
$\cup$ (union)
$\subseteq$ (inclusion)

A particularly useful operation is the union hull ($\uplus$):

Union Hull. Let $I_1 = <_1 a_1..b_1 >_1$ and $I_2 = <_2 a_2..b_2 >_2$ be two intervals. The union hull operation ($\uplus$) is defined as:

$$I_1 \uplus I_2 = \begin{cases} I_1 \cup I_2 & \text{if} \quad I_1 \cap I_2 \neq \emptyset \\ <_1 a_1..b_2 >_2 & \text{if} \quad \forall_{r_1 \in I_1} \forall_{r_2 \in I_2}\, r_1 < r_2 \\ <_2 a_2..b_1 >_1 & \text{if} \quad \forall_{r_1 \in I_1} \forall_{r_2 \in I_2}\, r_2 < r_1 \end{cases}$$

❑

In the case of closed intervals [*a*,*b*] and [*c*,*d*]:

$$[a,b] \uplus [c,d] = [min(a,c), max(b,d)]$$

Representation of Continuous Domains

Interval Operations and Basic Functions

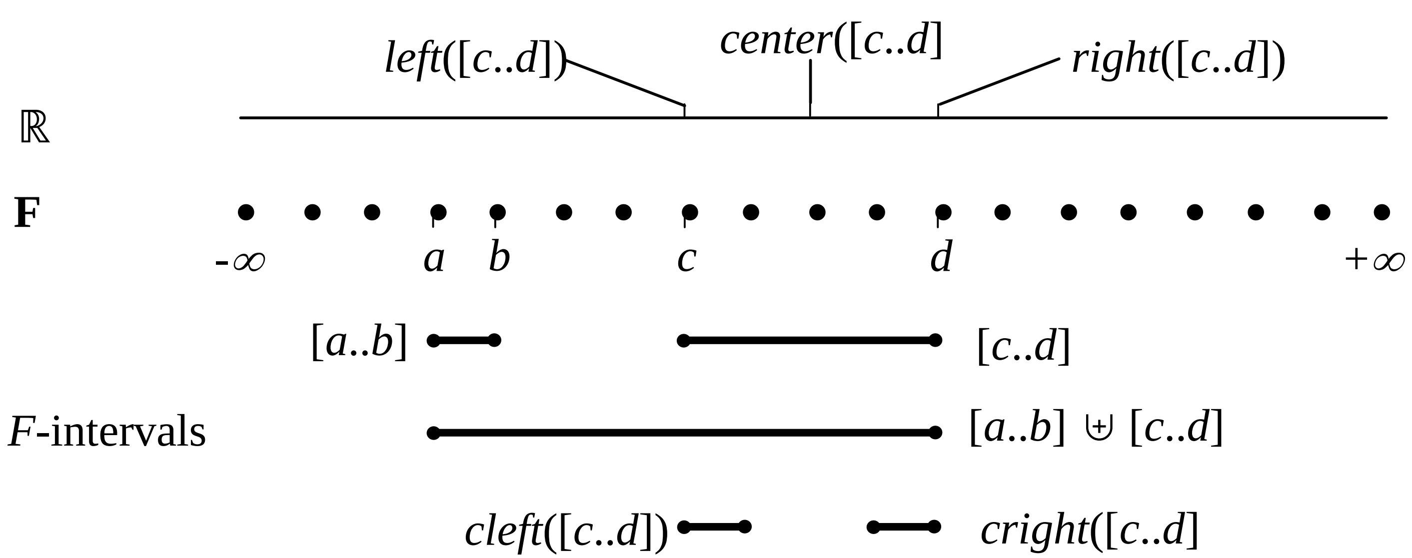
Interval Basic Functions. Let $[a..b]$ be a closed interval. The following basic functions return a real value and are defined as:

$$left([a..b]) = a \qquad right([a..b]) = b$$

$$center([a..b]) = (a+b)/2 \qquad width([a..b]) = b-a$$

Let $[a..b]$ be a closed F-interval. The following basic functions return a canonical F-interval and are defined as:

$$cleft([a..b]) = \begin{cases} [a] & \text{if } a=b \\ [a..a^{+}] & \text{if } a<b \end{cases} \qquad cright([a..b]) = \begin{cases} [b] & \text{if } a=b \\ [b^{-}..b] & \text{if } a<b \end{cases}$$ ❑



Representation of Continuous Domains

Interval Approximations

For any real number r we will denote by:

$\lfloor r \rfloor$ the largest F-number not greater than r $(\lfloor -\infty \rfloor = -\infty)$

$\lceil r \rceil$ the smallest F-number not smaller than r $(\lceil +\infty \rceil = +\infty)$

Interval Approximation. Let $IR=<a..b>$ be a real interval. The interval approximation of IR, denoted $I_{apx}(IR)$, is the smallest F-interval including IR ($IR \subseteq I_{apx}(IR)$):

$$I_{apx}(IR)=[\lfloor a \rfloor .. \lceil b \rceil].$$

In the special case where IR is a single real $\{r\}=[r..r]$ then $I_{apx}(IR)=[\lfloor r \rfloor .. \lceil r \rceil]$. ❑

Set Approximation. Let SR be a set of real values defined by the union of n real intervals ($SR=IR_1 \cup \ldots \cup IR_n$). The set approximation of SR, denoted $S_{apx}(SR)$, is the set defined by the union of the n corresponding interval approximations:

$$S_{apx}(SR) = I_{apx}(IR_1) \cup \ldots \cup I_{apx}(IR_n)$$ ❑

Hull Approximation. Let SR be a set of real values defined by the union of n real intervals ($SR=IR_1 \cup \ldots \cup IR_n$). The hull approximation of SR, denoted $I_{hull}(SR)$, is the F-interval defined by:

$$I_{hull}(SR) = I_{apx}(IR_1) \uplus \ldots \uplus I_{apx}(IR_n)$$ ❑

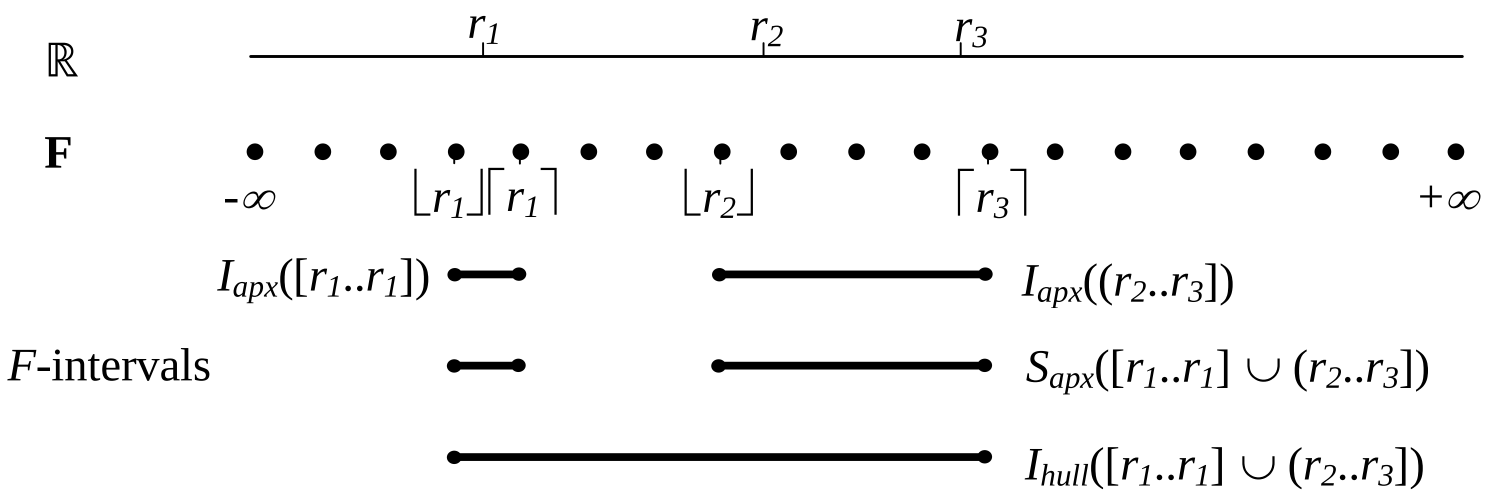
Representation of Continuous Domains

Interval Approximations

For any real number r we will denote by:

$\lfloor r \rfloor$ the largest F-number not greater than r $(\lfloor -\infty \rfloor = -\infty)$

$\lceil r \rceil$ the smallest F-number not smaller than r $(\lceil +\infty \rceil = +\infty)$



Interval Arithmetic

Interval arithmetic is an extension of real arithmetic for intervals

Basic Interval Arithmetic Operators

The basic operators are redefined for intervals:

the result is the set obtained by applying the operator to any pair of reals from the interval operands

Basic Interval Arithmetic Operators. Let I_1 and I_2 be two real intervals (bounded and closed). The basic arithmetic operations on intervals are defined by:

$$I_1 \Phi I_2 = \{ r_1 \Phi r_2 \mid r_1 \in I_1 \wedge r_2 \in I_2 \} \quad \text{with} \quad \Phi \in \{+,-,\times,/\}$$

except that I_1/I_2 is not defined if $0 \in I_2$. ❑

Algebraic rules may be defined to evaluate any basic operation on intervals in terms of formulas for its bounds

Evaluation Rules for the Basic Operators. Let $[a..b]$ and $[c..d]$ be two real intervals (bounded and closed):

$$[a..b] + [c..d] = [a+c..b+d] \qquad [a..b] - [c..d] = [a-d..b-c]$$

$$[a..b] \times [c..d] = [\min(ac,ad,bc,bd)..\max(ac,ad,bc,bd)]$$

$$[a..b] \,/\, [c..d] = [a..b] \times [1/d..1/c] \qquad \text{if } 0 \notin [c..d]$$ ❑

Interval Arithmetic

Algebraic Properties

Most algebraic properties of real arithmetic also hold for interval arithmetic: the distributive law is an exception

Algebraic Properties of the Basic Operators. Let I_1, I_2, I_3 and I_4 be real intervals (bounded and closed). The following algebraic properties hold for the basic interval operations:

Commutativity:	$I_1+I_2=I_2+I_1$	(interval addition)
	$I_1\times I_2=I_2\times I_1$	(interval multiplication)
Associativity:	$(I_1+I_2)+I_3=I_1+(I_2+I_3)$	(interval addition)
	$(I_1\times I_2)\times I_3=I_1\times(I_2\times I_3)$	(interval multiplication)
Neutral Element:	$I_1+[0..0]=I_1$	(interval addition)
	$I_1\times[1..1]=I_1$	(interval multiplication)
Subdistributivity:	$I_1\times(I_2+I_3)\subseteq I_1\times I_2+I_1\times I_3$	
Inclusion Monotonicity:	$I_1\subseteq I_3 \wedge I_2\subseteq I_4 \Rightarrow I_1\Phi I_2 \subseteq I_3\Phi I_4$	

(with: $\Phi\in\{+,-,\times,/\}$ and $I_3\Phi I_4$ defined) ❑

Inclusion monotonicity is an important new concept

Interval Arithmetic

Algebraic Properties

Example of Subdistributivity:

$$I_1\times(I_2+I_3) \subseteq I_1\times I_2+I_1\times I_3$$

$I_1=[0..1]$

$I_2=[2..3]$

$I_3=[-2..-1]$

$$[0..1]\times(\underbrace{[2..3]+[-2..-1]}) \quad \underbrace{[0..1]\times[2..3]}+\underbrace{[0..1]\times[-2..-1]}$$

$$\underbrace{[0..1] \times [0..2]} \quad \underbrace{[0..3] + [-2..0]}$$

$$[0..2] \subseteq [-2..3]$$

Example of Inclusion monotonicity:

(the same operations with smaller domains)

$$I_1\times(I_2+I_3) \subseteq I_1\times I_2+I_1\times I_3$$

$I_1=[0.5..1]$

$I_2=[2..2.5]$

$I_3=[-2..-1]$

$$[0.5..1]\times(\underbrace{[2..2.5]+[-2..-1]}) \quad \underbrace{[0.5..1]\times[2..2.5]}+\underbrace{[0.5..1]\times[-2..-1]}$$

$$\underbrace{[0.5..1] \times [0..1.5]} \quad \underbrace{[1..2.5] + [-2..-0.5]}$$

$$[0..1.5] \subseteq [-1..2]$$

Interval Arithmetic

Safe Evaluation

In interval arithmetic computations of the correct real values must be always within the bounds of the resulting interval

Outward rounding forces the result of any basic interval arithmetic operation to be the interval approximation of the correct real interval (obtained with infinite precision)

Outward Rounding Evaluation Rules of the Basic Operators. Let $[a..b]$ and $[c..d]$ be two F-intervals (bounded and closed):

$$[a..b] + [c..d] = [\lfloor a+c \rfloor .. \lceil b+d \rceil] \qquad [a..b] - [c..d] = [\lfloor a-d \rfloor .. \lceil b-c \rceil]$$

$$[a..b] \times [c..d] = [\min(\lfloor ac \rfloor, \lfloor ad \rfloor, \lfloor bc \rfloor, \lfloor bd \rfloor)..\max(\lceil ac \rceil, \lceil ad \rceil, \lceil bc \rceil, \lceil bd \rceil)]$$

$$[a..b] \,/\, [c..d] = [a..b] \times [\lfloor 1/d \rfloor .. \lceil 1/c \rceil] \qquad \text{if } 0 \notin [c..d]$$

❑

If Φ is a basic interval arithmetic operator then Φ_{apx} denotes the corresponding outward evaluation rule: $\Phi_{apx}(I_1,\ldots,I_m)=I_{apx}(\Phi(I_1,\ldots,I_m))$

Interval Arithmetic

Safe Evaluation

In interval arithmetic computations the correct real values must be always within the bounds of the resulting interval

The correctness of the interval arithmetic computations is guaranteed by the inclusion monotonicity property:

> if the correct real values are within the operand intervals then the correct real values resulting from any interval arithmetic operation must also be within the resulting interval.

The computation of a successive composition of basic arithmetic operations over real intervals preserve the correct real values within the final resulting interval

Interval Arithmetic

Extended Interval Arithmetic

Extensions on the definition of the division operator:

allow division by an interval containing 0

if $c<0<d$ then $[a,b]/[c,d]=[a,b]/[c,0^-] \cup [a,b]/[0^+,d]$

$[1,2]/[-1,1] = [1,2]/[-1,0^-] \cup [1,2]/[0^+,1]$

$[-\infty,-1] \cup [1,+\infty]$

Extensions on the real intervals allowed as arguments:

allow open intervals and infinite bounds

$(-\infty,-1]+[-1,3] = (-\infty,2]$

$(-\infty,-1]+[-1,+\infty] = (-\infty,+\infty]$

Extensions on the set of basic interval operators:

allow other elementary functions (*exp*, *ln*, *power*, *sin*, *cos*…)

$exp([a,b]) = [exp(a),exp(b)]$

Interval Functions

Interval Expressions and their Evaluation

Real and Interval Expressions. An expression E is an inductive structure defined in the following way:

(i) a constant is an expression;
(ii) a variable is an expression;
(iii) if $E_1,\ldots,E_m$ are expressions and Φ is a m-ary basic operator then $\Phi(E_1,\ldots,E_m)$ is an expression;

A real expression is an expression with real constants, real valued variables and real operators. An interval expression is an expression with interval constants, interval valued variables and interval operators. ❑

If x_1, x_2 and x_3 are real valued variables then $(x_1+x_2)\times(x_3-\pi)$ is a real expression with three binary real operators (+, × and -) and a real constant (π).

If X_1 and X_2 are interval valued variables then $(X_1+cos([0..\pi]\times X_2))$ is an interval expression with two binary interval operators (+ and ×), a unary interval operator (*cos*) and an interval constant ([0..π]).

Interval Functions

Interval Expressions and their Evaluation

Interval arithmetic provides a safe method for evaluating an interval expression:

replace each variable by its interval domain;
apply recursively the basic operator evaluation rules

Evaluation of an Interval Expression. Let F be the n-ary interval function represented by the interval expression F_E, and B an n-ary R-box. The interval arithmetic evaluation of F_E wrt B is an interval function recursively defined as:

$$F_E(B) = \begin{cases} I_{apx}(I) & \text{if } F_E \equiv I & (I \text{ is an interval constant}) \\ I_{apx}(B[X_i]) & \text{if } F_E \equiv X_i & (X_i \text{ is an interval variable}) \\ \Phi_{apx}(E_1(B),\ldots, E_m(B)) & \text{if } F_E \equiv \Phi(E_1,\ldots,E_m) & (\Phi \text{ is an interval operator}) \end{cases}$$ ❑

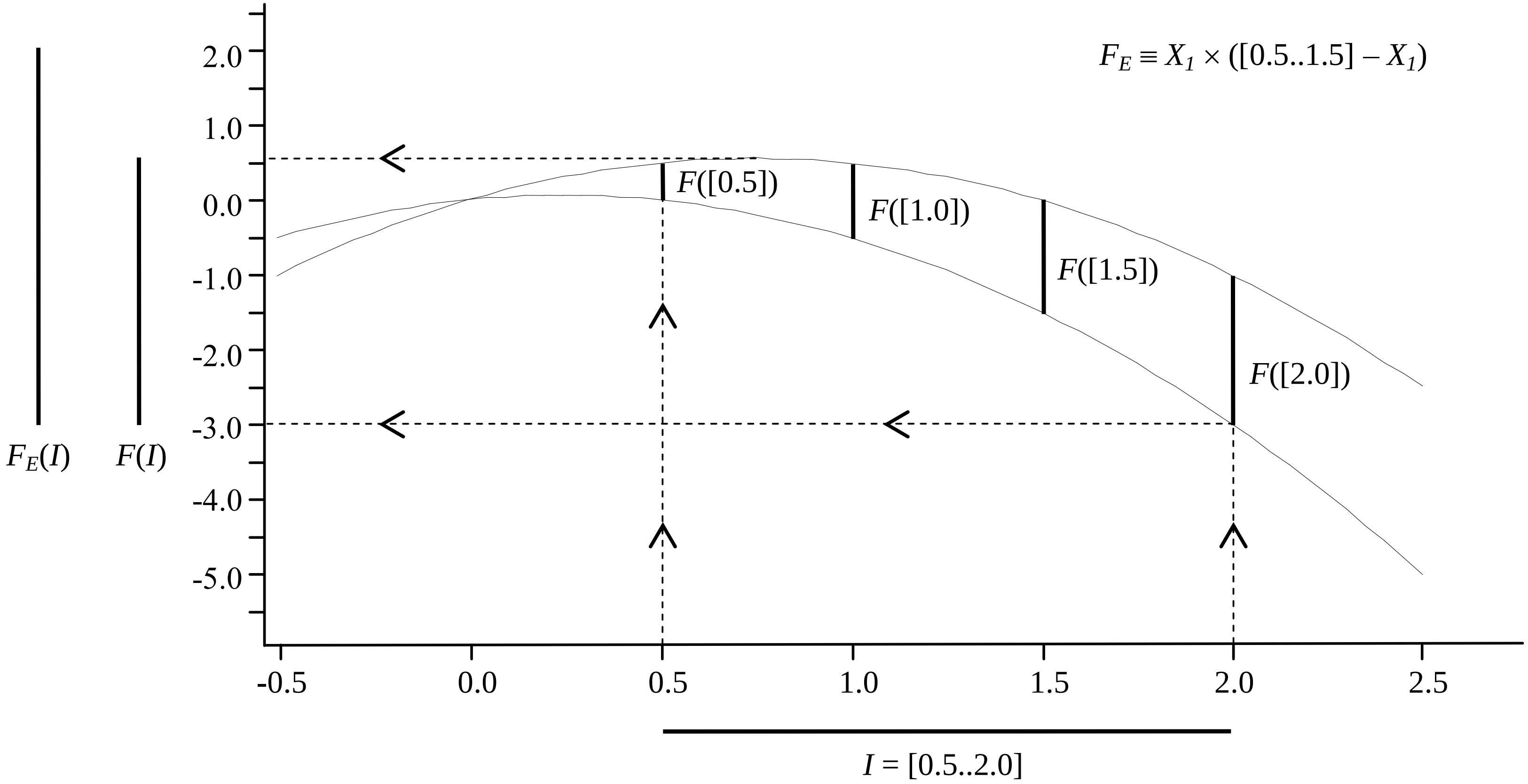
The interval arithmetic evaluation of an interval expression provides a sound computation of the interval function represented by the expression

Interval Functions

Interval Expressions and their Evaluation

Soundness of the Interval Expression Evaluation. Let F_E be an interval expression representing the n-ary interval function F, and B an n-ary R-box. The interval arithmetic evaluation of F_E with respect to B is sound:

$$F(B) \subseteq F_E(B)$$ ❑



Interval Functions

Interval Extensions

> **Interval Extension of a Real Function.** Let f be an n-ary real function with domain D_f, and F an n-ary interval function. The interval function F is an interval extension of the real function f iff:
>
> $$\forall_{<r_1,\ldots,r_n>\in D_f}\ f(<r_1,\ldots,r_n>) \in F(<[r_1..r_1],\ldots,[r_n..r_n]>)$$ ❑

If *F* is an interval extension of *f* then each real value mapped by *f* must lie within the interval mapped by *F* when the argument is the corresponding box of degenerate intervals

Consequently, *F* provides a sound evaluation of *f* in the sense that the correct real value is not lost

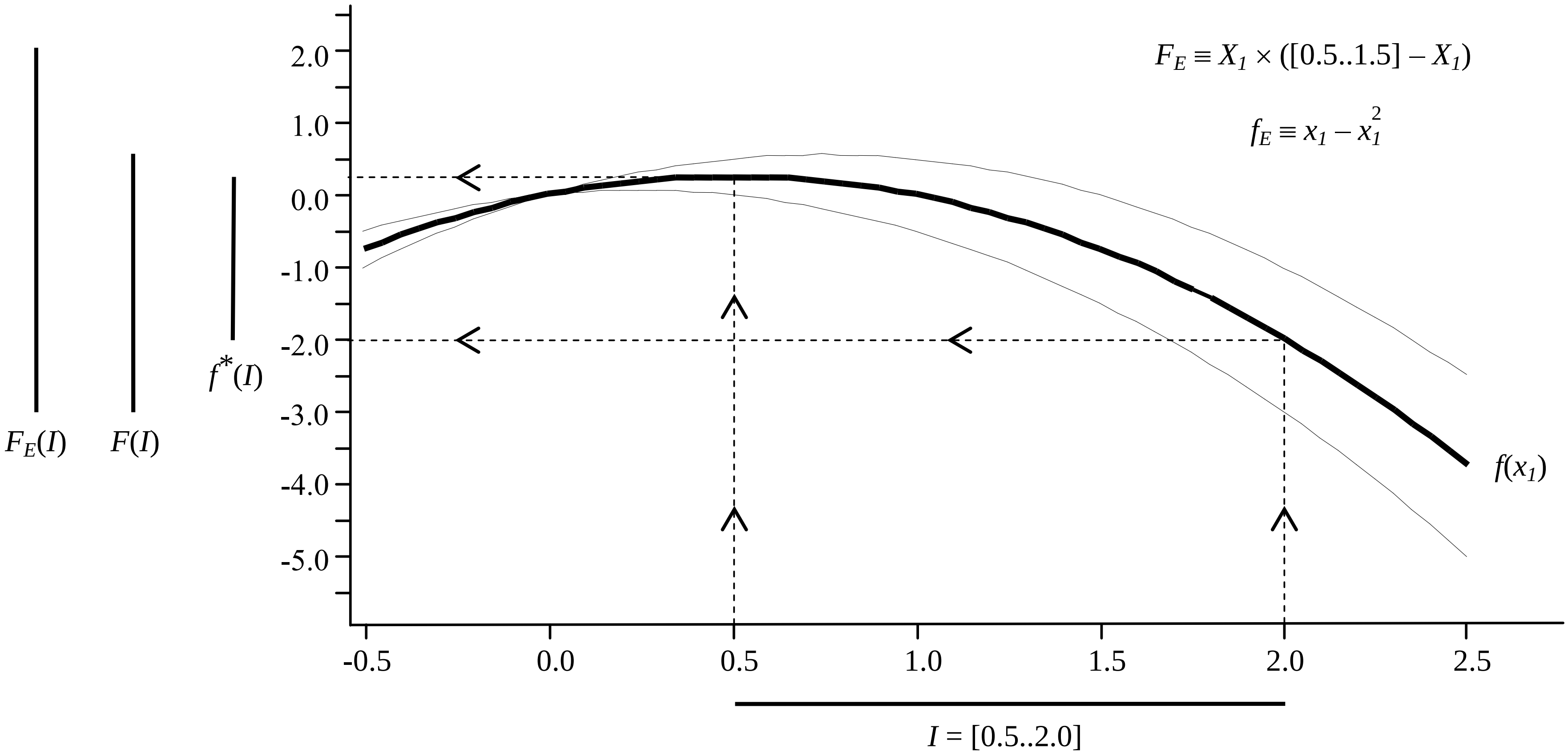
The interval arithmetic evaluation of any expression representing an interval extension of a real function provides a sound evaluation for its range and is itself an interval extension of the real function

Interval Functions

Interval Extensions

Soundness of the Evaluation of an Interval Extension. Let F be an interval extension of an n-ary real function f, F_E an interval expression representing F, and B be n-ary R-box. Then, both $F(B)$ and $F_E(B)$, enclose the range of f over B:

$$f^*(B) \subseteq F(B) \subseteq F_E(B)$$ ❑



Interval Functions

Interval Extensions

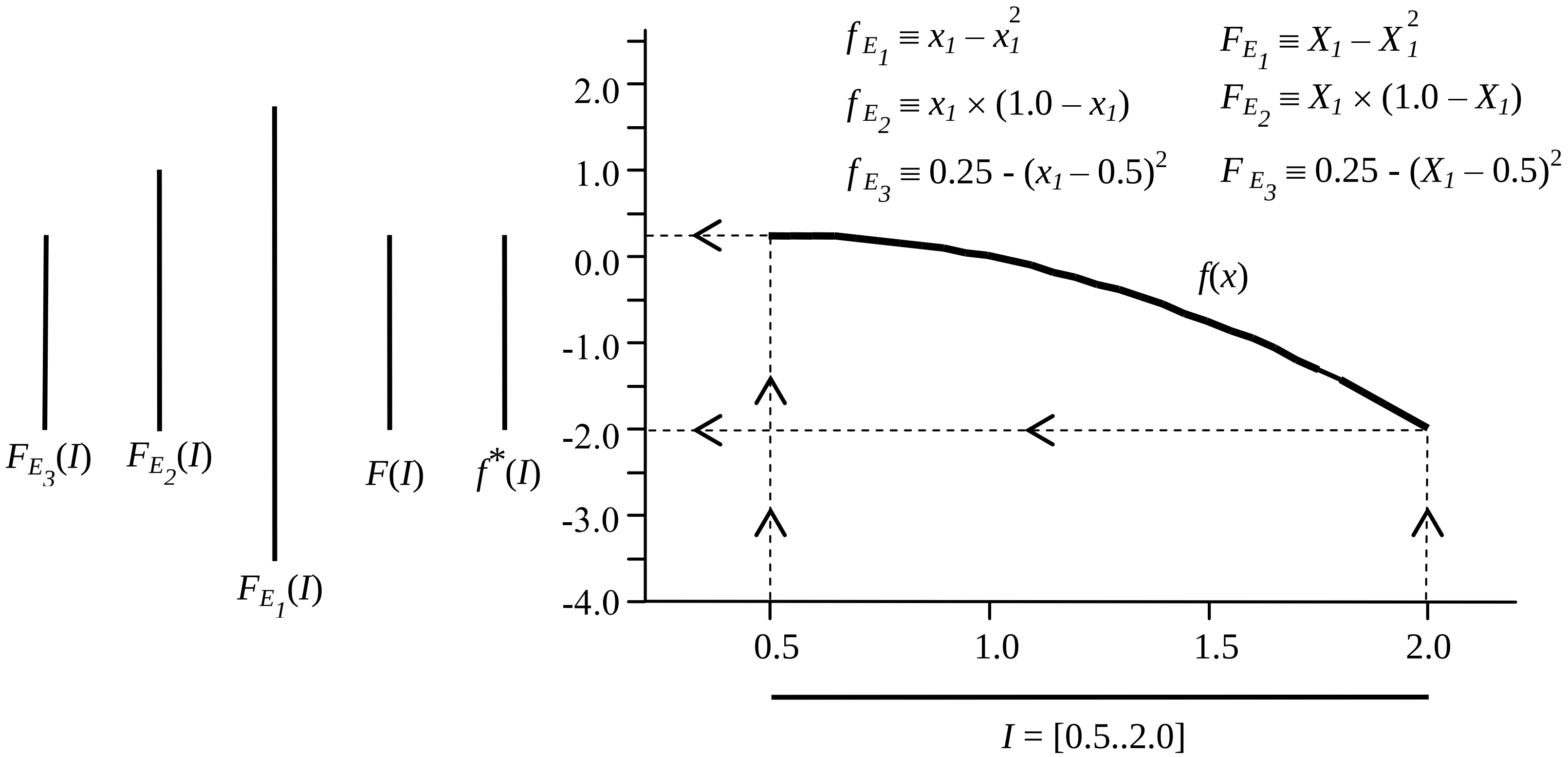
Natural Interval Expression. If f_E is a real expression representing the real function f, then its natural interval expression F_n is obtained by replacing in f_E: each real variable x_i by an interval variable X_i; each real constant k by the real interval $[k..k]$, and each real operator by the corresponding interval operator. ❑

Natural Interval Extension. Let f_E be a real expression representing the real function f, and F_n be the natural interval expression of f_E. The interval function F represented by F_n is the smallest interval enclosure for the range of f and the interval arithmetic evaluation of F_n is an interval extension of f denominated Natural interval extension w.r.t. f_E. ❑

Several equivalent real expressions may represent the same real function f. Consequently, the natural interval extensions with respect to these equivalent real expressions are all interval extensions of f.

Interval Functions

Interval Extensions



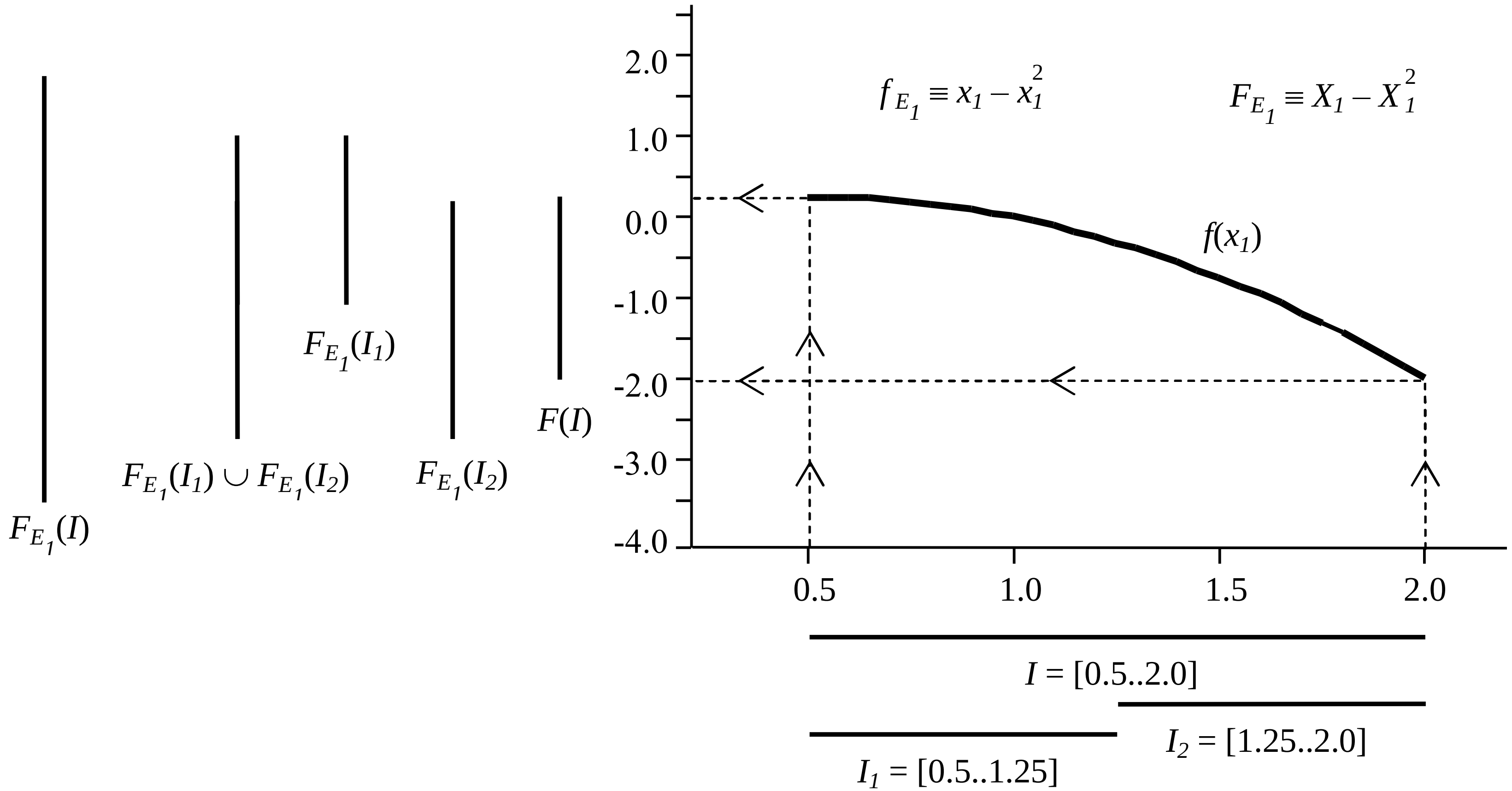
Intersection of Interval Extensions. Let F_1 and F_2 be two n-ary interval functions and B an n-ary R-box. Let F be an n-ary interval function defined by: $F(B)=F_1(B)\cap F_2(B)$.
If F_1 and F_2 are interval extensions of the real function f, then F is also an interval extension of f. ❑

Interval Functions

Interval Extensions

Decomposed Evaluation of an Interval Extension. Let F be an interval extension of the n-ary real function f, and F_E an interval expression representing F. Let B, B_1 and B_2 be n-ary R-boxes. If $B=B_1 \cup B_2$ then:

$$F(B) \subseteq F_E(B_1) \cup F_E(B_2) \subseteq F_E(B)$$ ❑



Interval Functions

Interval Extensions

Dependency Problem. In the interval arithmetic evaluation of an interval expression, each occurrence of the same variable is treated as a different variable. The dependency between the different occurrences of a variable in an expression is lost. ❑

No Overestimation Without Multiple Variable Occurrences. Let F_E be an interval expression representing the n-ary interval function F, and B an n-ary R-box. If F_E is an interval expression in which each variable occurs only once then:

$F(B) = F_E(B)$ (w/ exact interval operators and infinite precision arithmetic evaluation) ❑