Interval Constraints Overview

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Lecture 1: Interval Constraints Overview

- **Continuous Constraint Satisfaction Problems**
- **Continuous Constraint Reasoning**
  - Representation of Continuous Domains
  - Pruning and Branching
- **Solving Continuous CSPs**
  - Constraint Propagation
  - Consistency Criteria
- **Practical Examples**
- **Course Structure**

Constraint Reasoning

Continuous CSP (CCSP):

Constraint Satisfaction Problem (CSP):

set of variables

set of domains → Intervals of reals [a,b]

set of constraints → Numeric (=,≤,≥)

Solution: → Many

assignment of values which satisfies all the constraints

GOAL Find Solutions;
Find an enclosure of the solution space

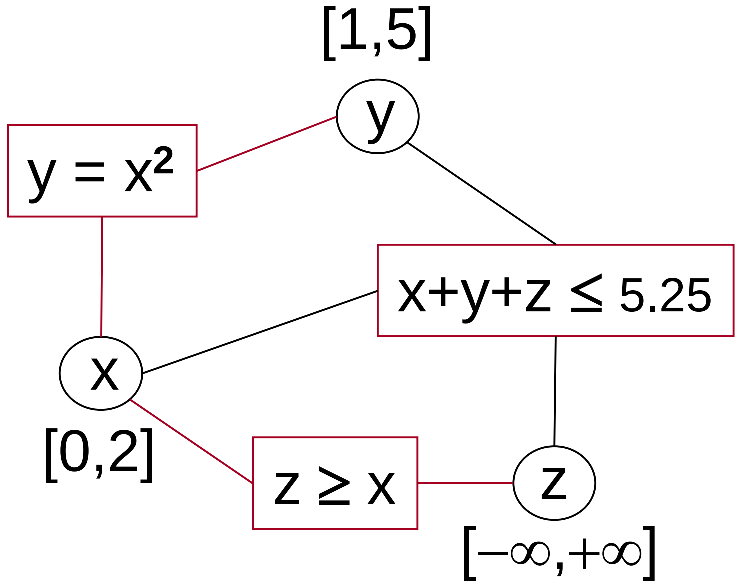
Constraint Reasoning

Continuous Constraint Satisfaction Problem (CCSP):

Interval Domains

Numerical Constraints

Many Solutions



Solution:

$x=1,\ y=1,\ z=1$

...

$x=1,\ y=1,\ z=3.25$

...

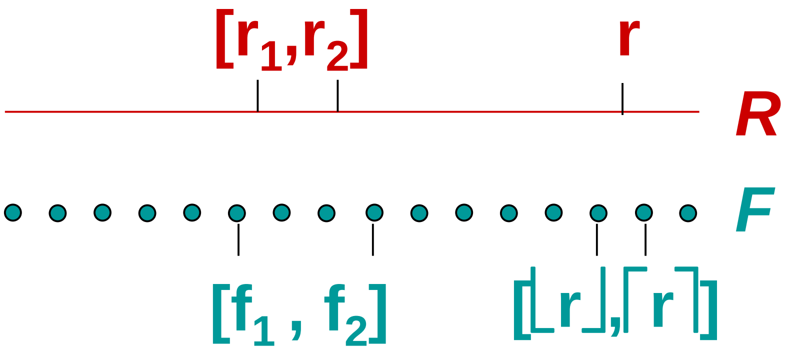
assignment of values which satisfies all the constraints

GOAL Find solutions;

Find an enclosure of the solution space

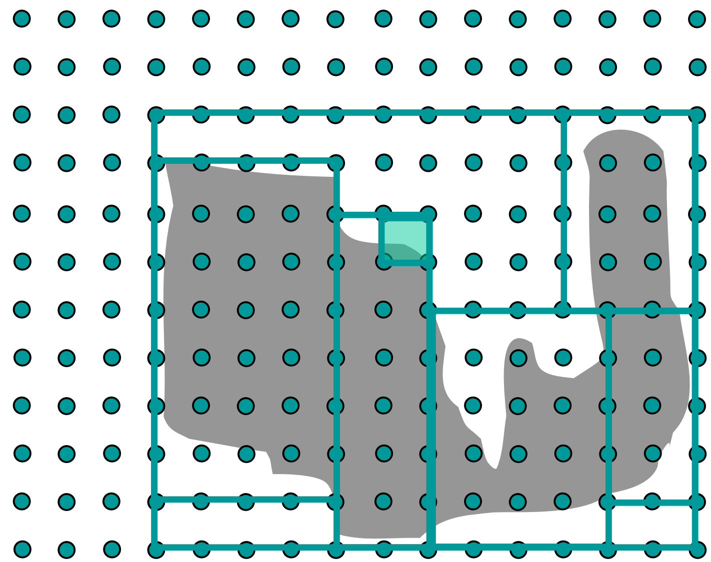
Representation of Continuous Domains

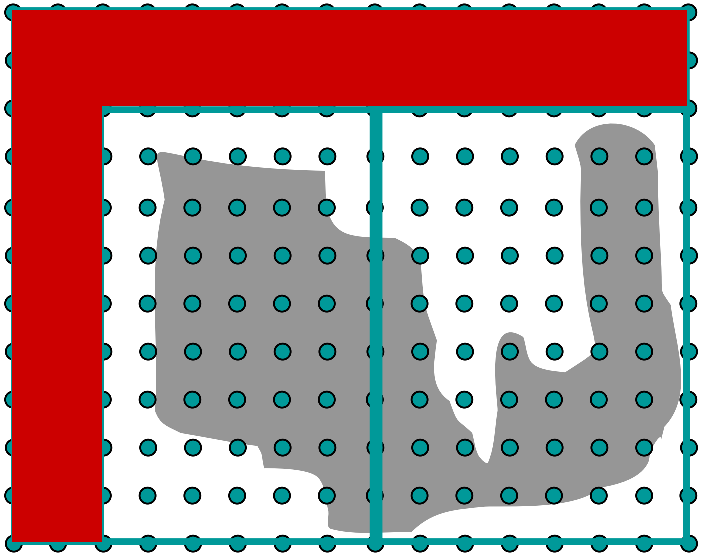
***F*-interval**



***F*-box**

Canonical solution





Solving CCSPs:

Branch and Prune algorithms

constraint propagation

box split

Safe Narrowing Functions

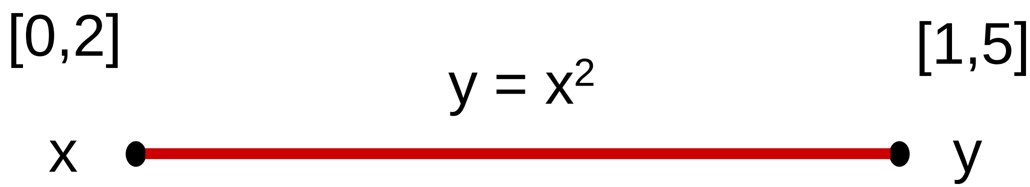
Strategy for
- isolate canonical solutions
- provide an enclosure of the solution space

depends on a consistency requirement

Constraint Reasoning (vs Simulation)

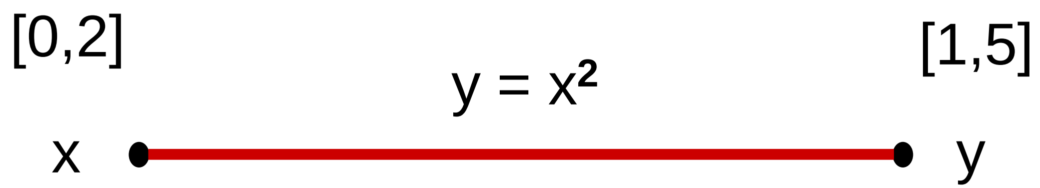
Represents uncertainty as intervals of possible values

Uses safe methods for narrowing the intervals accordingly to the constraints of the model



Simulation:		0	0 no
	x≤1?	1	1
		2	4 y≥4?
Constraint Reasoning:		[1,2]	[1,4]

How to narrow the domains?



Safe methods are based on Interval Analysis techniques

$$\forall_{x\in[a,b]}\; x^2 \in [a,b]^2 = \begin{cases} [0, \lceil \max(a^2,b^2) \rceil] & \text{if } a \le 0 \le b \\ [\lfloor \min(a^2,b^2) \rfloor, \lceil \max(a^2,b^2) \rceil] & \text{otherwise} \end{cases}$$

If $x \in [0,2]$

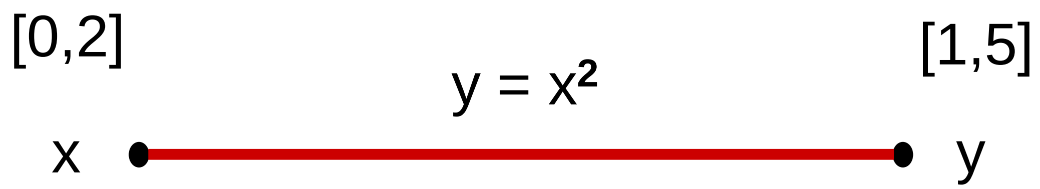
Then $y \in [0,2]^2 = [0, \lceil \max(0^2,2^2) \rceil] = [0,4]$

$\therefore\; y \in [1,5] \wedge y \in [0,4]$

$\therefore\; y \in [1,5] \cap [0,4]$

$\therefore\; y \in [1,4]$

How to narrow the domains?



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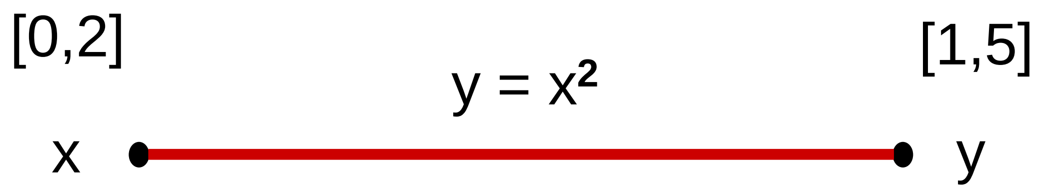
$$NF_{y=x^2}: Y' \leftarrow Y \cap X^2$$

How to narrow the domains?



Safe methods are based on Interval Analysis techniques

$y - x^2 = 0 \longrightarrow F(Y) = Y - [0,2]^2 \qquad F'(Y) = 1$

$$\forall_{y \in Y} \forall_{x \in [0,2]} \; y - x^2 = 0 \Rightarrow y \in N(Y) = c(Y) - \frac{F(c(Y))}{F'(Y)}$$

Interval Newton method

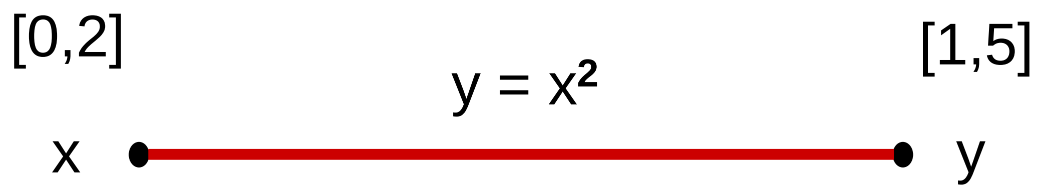
If $x \in [0,2]$ and $y \in [1,5]$

Then $y \in N([1,5]) = 3 - \dfrac{3 - [0,2]^2}{1} = [0,4]$

$\therefore \; y \in [1,5] \cap [0,4]$

$\therefore \; y \in [1,4]$

How to narrow the domains?



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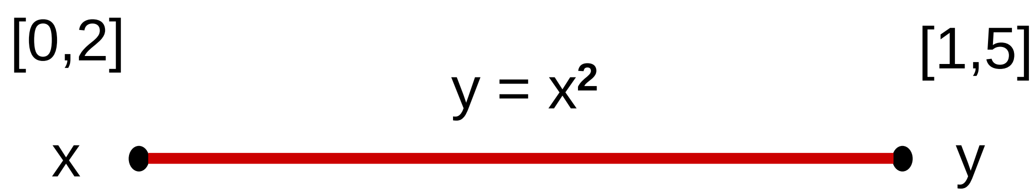
Interval Newton method

$$NF_{y=x^2}: Y' \leftarrow Y \cap \left(c(Y) - \frac{c(Y) - X^2}{1} \right)$$

How to narrow the domains?



Safe methods are based on Interval Analysis techniques

contractility | correctness

$NF_{y=x^2}$: $Y' \leftarrow Y \cap X^2$ $\quad Y' \subseteq Y \quad \forall_{y\in Y}\, y \notin Y' \Rightarrow \neg\exists_{x\in X}\, y=x^2$

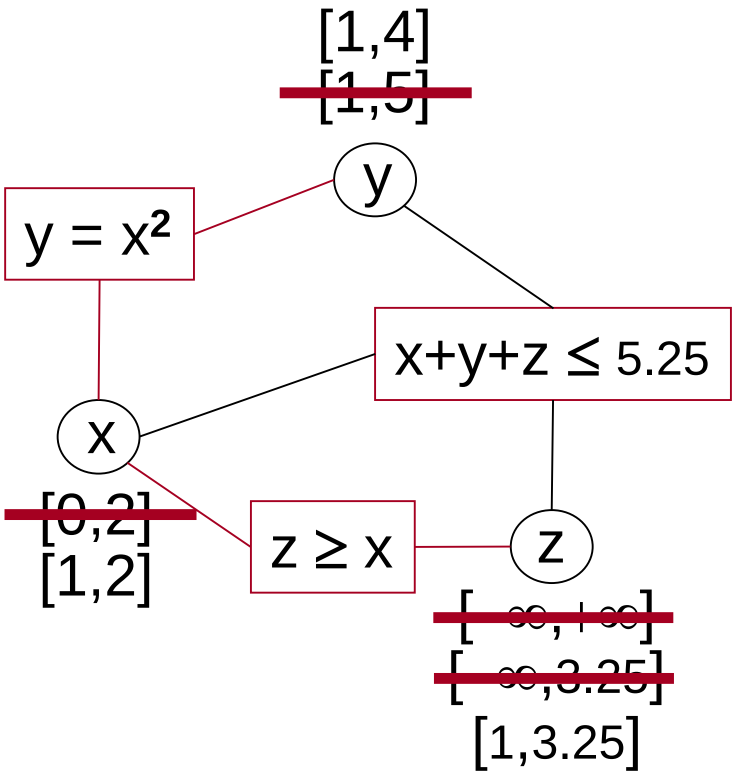
$$NF_{y=x^2}: Y' \leftarrow Y \cap \left(c(Y) - \frac{c(Y) - X^2}{1} \right)$$

$NF_{y=x^2}$: $X' \leftarrow (X \cap -Y^{1/2}) \uplus (X \cap +Y^{1/2})$ $\quad X' \subseteq X \quad \forall_{x\in X}\, x \notin X' \Rightarrow \neg\exists_{y\in Y}\, y=x^2$

$$NF_{y=x^2}: X' \leftarrow X \cap \left(c(X) - \frac{Y - c(X)^2}{-2X} \right)$$

Solving a Continuous Constraint Satisfaction Problem

Constraint Propagation



$NF_{y=x^2}: Y' \leftarrow Y \cap X^2$

$NF_{y=x^2}: X' \leftarrow (X \cap -Y^{½}) \uplus (X \cap +Y^{½})$

$NF_{x+y+z \leq 5.25}: X' \leftarrow X \cap ([-\infty, 5.25] - Y - Z)$

$NF_{x+y+z \leq 5.25}: Y' \leftarrow Y \cap ([-\infty, 5.25] - X - Z)$

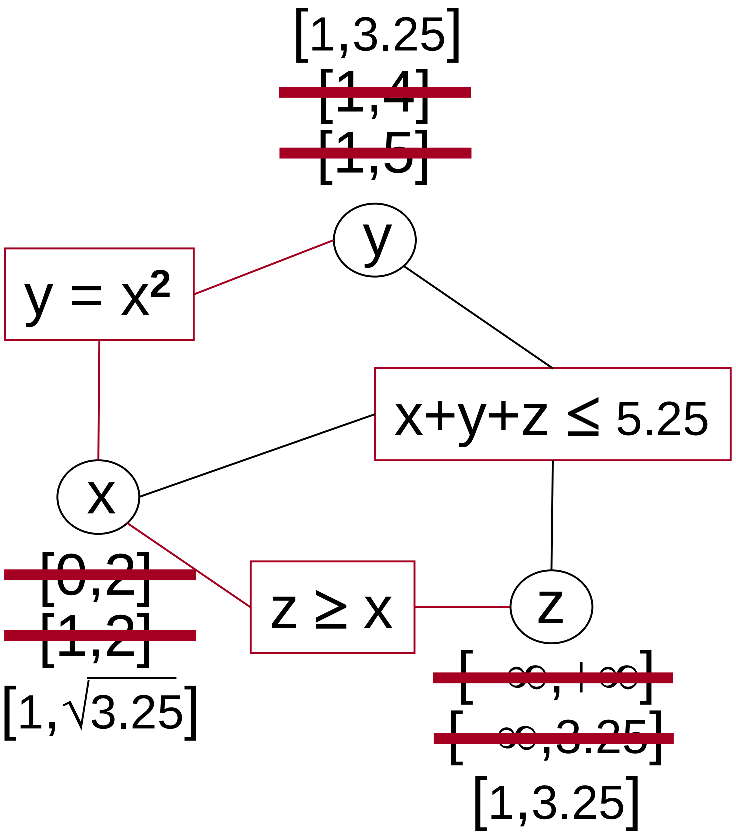
$NF_{x+y+z \leq 5.25}: Z' \leftarrow Z \cap ([-\infty, 5.25] - X - Y)$

$NF_{z \geq x}: X' \leftarrow X \cap (Z - [0, +\infty])$

$NF_{z \geq x}: Z' \leftarrow Z \cap (X + [0, +\infty])$

Solving a Continuous Constraint Satisfaction Problem

Constraint Propagation



√ $NF_{y=x^2}$: $Y' \leftarrow Y \cap X^2$

√ $NF_{y=x^2}$: $X' \leftarrow (X \cap -Y^{½}) \uplus (X \cap +Y^{½})$

√ $NF_{x+y+z \leq 5.25}$: $X' \leftarrow X \cap ([-\infty, 5.25] - Y - Z)$

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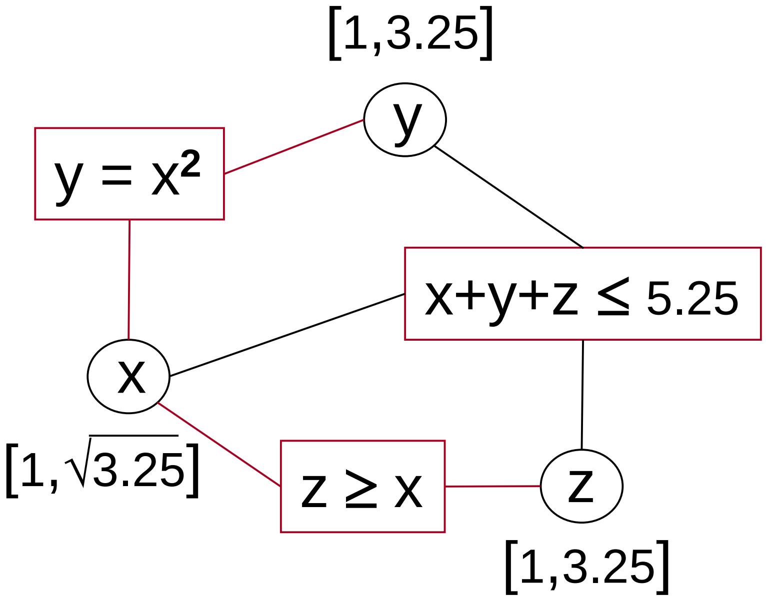
√ $NF_{x+y+z \leq 5.25}$: $Z' \leftarrow Z \cap ([-\infty, 5.25] - X - Y)$

√ $NF_{z \geq x}$: $X' \leftarrow X \cap (Z - [0, +\infty])$

√ $NF_{z \geq x}$: $Z' \leftarrow Z \cap (X + [0, +\infty])$

Solving a Continuous Constraint Satisfaction Problem

{ Constraint Propagation + Branching
Consistency Criterion



x	y	z	
1	1	1	√
1	1	3.25	√
$\sqrt{3.25}$			X
1.5	2.25	1.5	√

$y = x^2 \Rightarrow y = 3.25$

$x+y+z \leq 5.25 \Rightarrow z \leq \underbrace{2-\sqrt{3.25}}_{<\sqrt{3.25}}$

~~$z \geq x$~~

Solving a Continuous Constraint Satisfaction Problem

{ Constraint Propagation + Branching
{ Consistency Criterion

Local Consistency (2B-Consistency) ⟵ Constraint Propagation

⇣

Higher Order Consistencies (kB-Consistency) ⟵ Constraint Propagation + Branching

3B-Consistency: if 1 bound is fixed then the problem is Local Consistent

x	y	z			x	y	z
$[1,\sqrt{3.25}]$	[1,3.25]	[1,3.25]	not 3B-Consistent	⟵	$[\sqrt{3.25}]$	[1,3.25]	[1,3.25]
[1,1.5]	[1,2.25]	[1,3.25]	3B-Consistent		not Local Consistent		

Example:

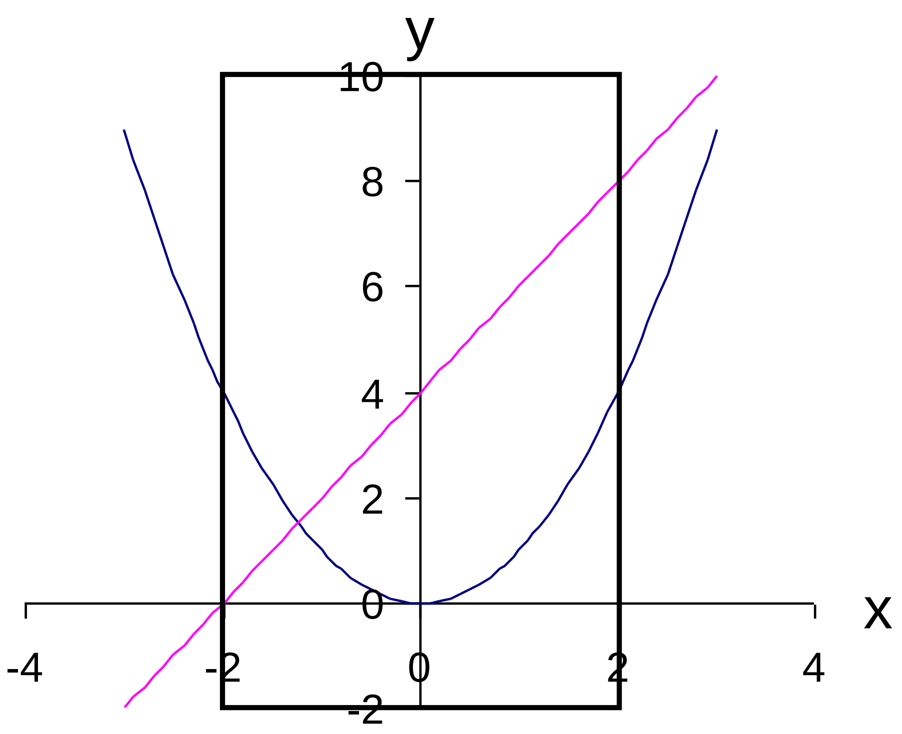
Variables: x, y

Domains: $[-2,2]\times[-2,10]$

Constraints:

$y = x^2$

$y \geq 2x + 4$



Constraint propagation

define set of narrowing functions:

$y = x^2$ → $NF_{y=x^2}: Y' \leftarrow Y \cap X^2$

$x = \pm y^{½}$ → $NF_{y=x^2}: X' \leftarrow (X \cap -Y^{½}) \uplus (X \cap +Y^{½})$

$y = 2x + [4,+\infty]$ → $NF_{y\geq 2x+4}: Y' \leftarrow Y \cap (2X + [4,+\infty])$

$x = ½y - [2,+\infty]$ → $NF_{y\geq 2x+4}: X' \leftarrow X \cap (½Y - [2,+\infty])$

Example:

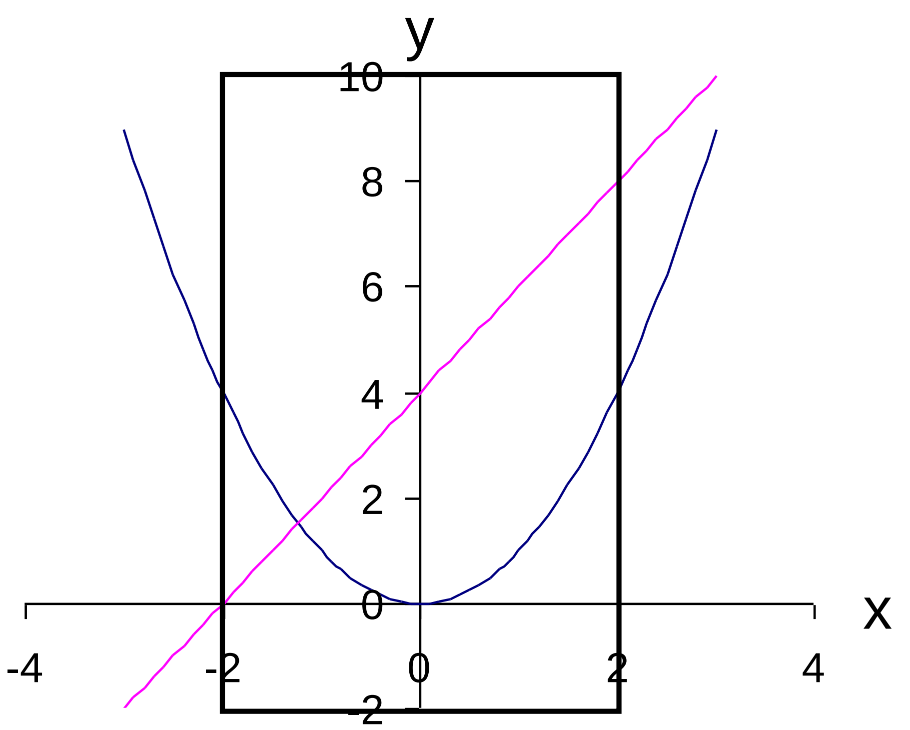
Variables: x, y

Domains: [-2,2]×[-2,10]

Constraints:

$y = x^2$

$y \geq 2x + 4$



Constraint propagation

apply the narrowing functions to prune box: [-2,2]×[-2,10]

$[-2,2]\times([-2,10] \cap [-2,2]^2)$ ← $NF_{y=x^2}$: $Y' \leftarrow Y \cap X^2$

$[-2,2]\times([-2,10] \cap [0,4])$ — $NF_{y=x^2}$: $X' \leftarrow (X \cap -Y^{1/2}) \uplus (X \cap +Y^{1/2})$

$[-2,2]\times[0,4]$ — $NF_{y\geq 2x+4}$: $Y' \leftarrow Y \cap (2X+[4,+\infty])$

$NF_{y\geq 2x+4}$: $X' \leftarrow X \cap (½Y-[2,+\infty])$

Example:

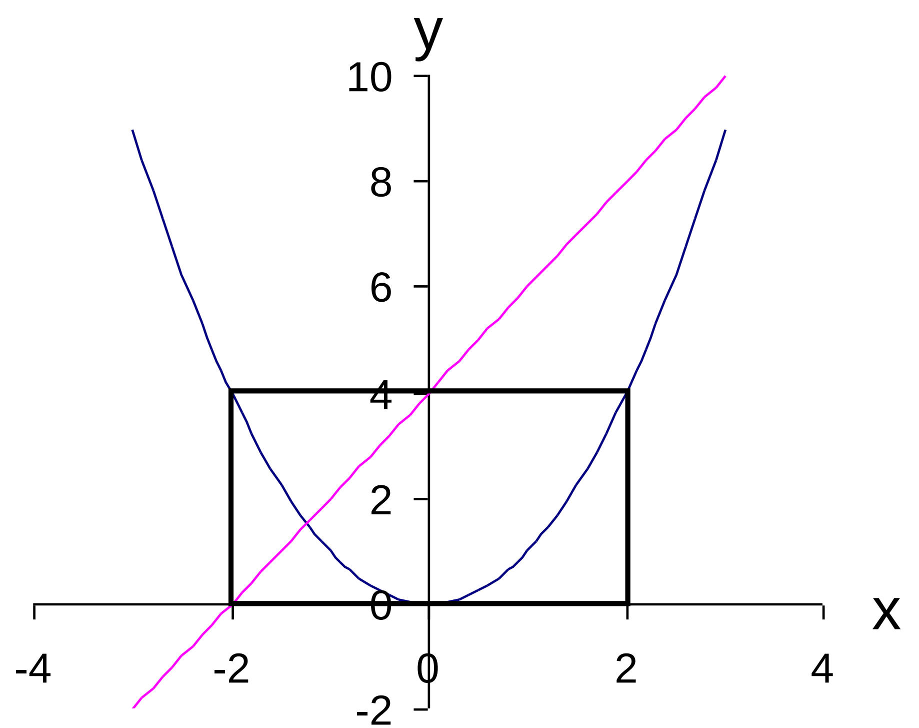
Variables: x, y

Domains: $[-2,2] \times [-2,10]$

Constraints:

$y = x^2$

$y \geq 2x + 4$



Constraint propagation

apply the narrowing functions to prune box: $[-2,2] \times [0,4]$

$NF_{y=x^2}$: $Y' \leftarrow Y \cap X^2$

$NF_{y=x^2}$: $X' \leftarrow (X \cap -Y^{½}) \cup (X \cap +Y^{½})$

$([-2,2] \cap -[0,4]^{½}) \uplus ([-2,2] \cap [0,4]^{½}) \times [0,4]$

$([-2,2] \cap [-2,0]) \uplus ([-2,2] \cap [0,2]) \times [0,4]$

$[-2,2] \times [0,4]$

$NF_{y \geq 2x+4}$: $Y' \leftarrow Y \cap (2X + [4, +\infty])$

$NF_{y \geq 2x+4}$: $X' \leftarrow X \cap (½Y - [2, +\infty])$

Example:

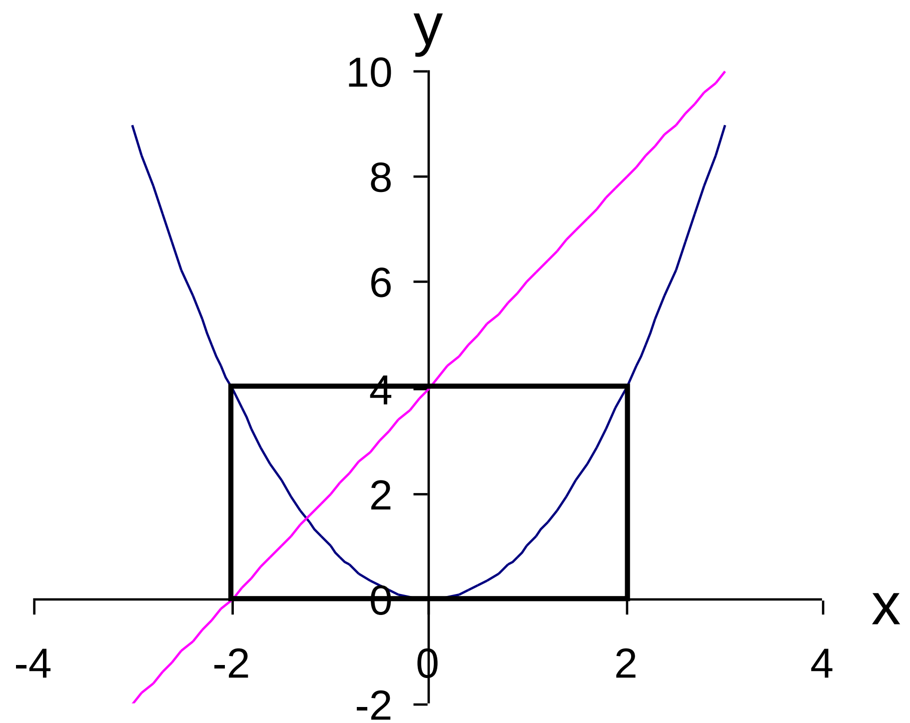
Variables: x, y

Domains: $[-2,2]\times[-2,10]$

Constraints:

$y = x^2$

$y \geq 2x + 4$



Constraint propagation

apply the narrowing functions to prune box: $[-2,2]\times[0,4]$

$NF_{y=x^2}$: $Y' \leftarrow Y \cap X^2$

$NF_{y=x^2}$: $X' \leftarrow (X \cap -Y^{1/2}) \cup (X \cap +Y^{1/2})$

$NF_{y\geq 2x+4}$: $Y' \leftarrow Y \cap (2X+[4,+\infty])$

$NF_{y\geq 2x+4}$: $X' \leftarrow X \cap (\tfrac{1}{2}Y - [2,+\infty])$

$[-2,2]\times([0,4]\cap(2[-2,2]+[4,+\infty]))$

$[-2,2]\times([0,4]\cap([-4,4]+[4,+\infty]))$

$[-2,2]\times([0,4]\cap[0,+\infty])$

$[-2,2]\times[0,4]$

Example:

Variables: x, y

Domains: $[-2,2]\times[-2,10]$

Constraints:

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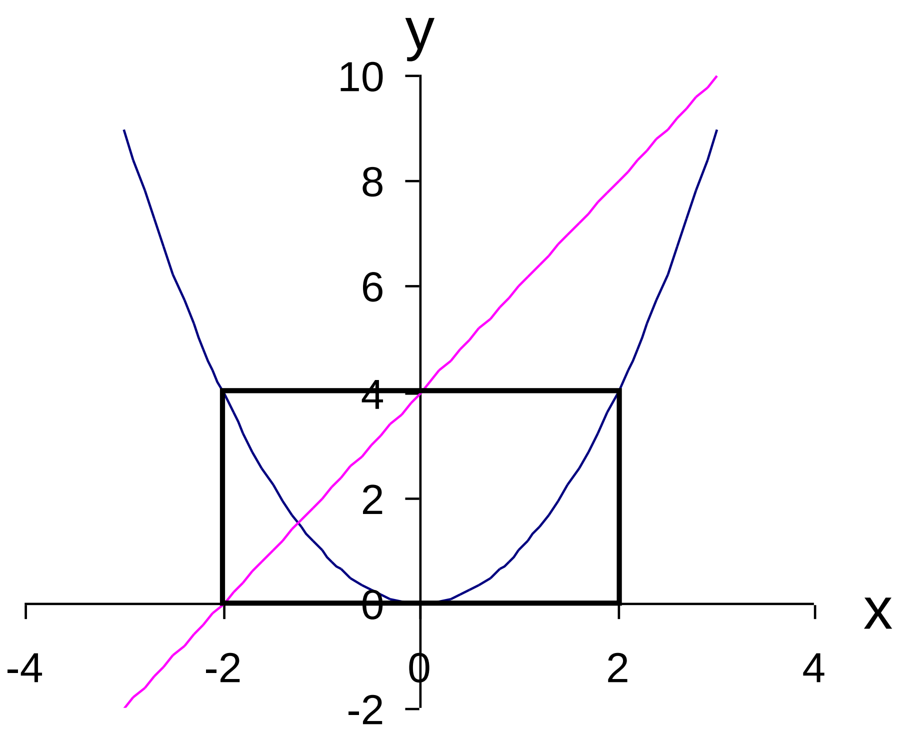

Constraint propagation

apply the narrowing functions to prune box: $[-2,2]\times[0,4]$

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$NF_{y\geq 2x+4}$: $Y' \leftarrow Y \cap (2X+[4,+\infty])$

$NF_{y\geq 2x+4}$: $X' \leftarrow X \cap (\tfrac{1}{2}Y-[2,+\infty])$

$([-2,2] \cap (\tfrac{1}{2}[0,4]-[2,+\infty]))\times[0,4]$

$([-2,2] \cap ([0,2]-[2,+\infty]))\times[0,4]$

$([-2,2] \cap [-\infty,0])\times[0,4]$

$[-2,0]\times[0,4]$ ←

Example:

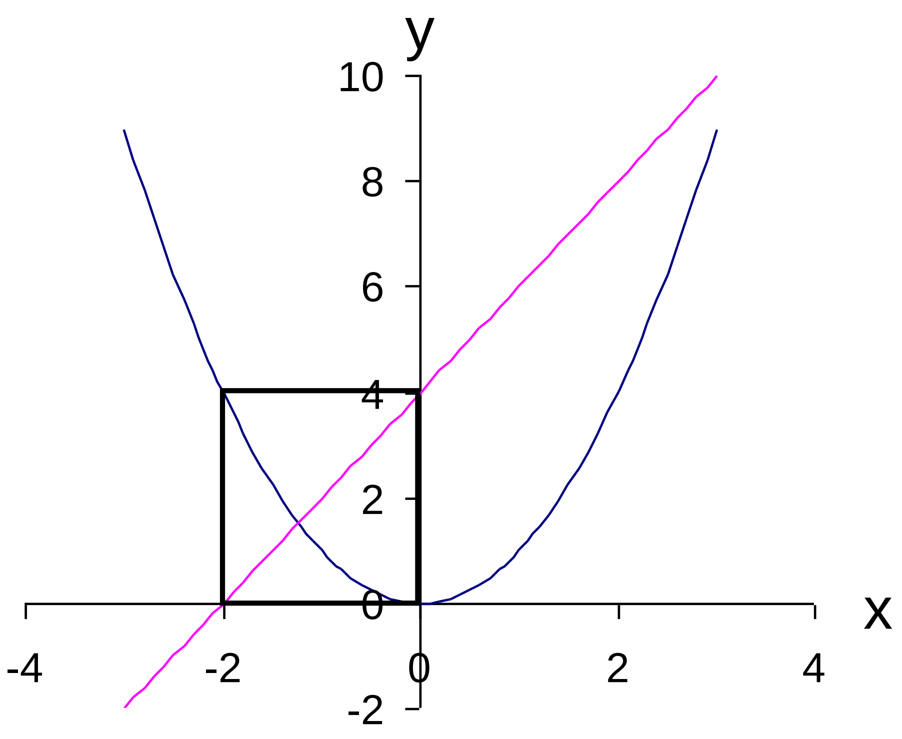
Variables: x, y

Domains: $[-2,2]\times[-2,10]$

Constraints:

$y = x^2$

$y \geq 2x + 4$



Constraint propagation

obtained the box: $[-2,0]\times[0,4]$ **(fixed point)**

$NF_{y=x^2}$: $Y' \leftarrow Y \cap X^2$

$NF_{y=x^2}$: $X' \leftarrow (X \cap -Y^{1/2}) \cup (X \cap +Y^{1/2})$

$NF_{y\geq 2x+4}$: $Y' \leftarrow Y \cap (2X+[4,+\infty])$

$NF_{y\geq 2x+4}$: $X' \leftarrow X \cap (\frac{1}{2}Y-[2,+\infty])$

Example:

Variables: x, y

Domains: [-2,2]×[-2,10]

Constraints:

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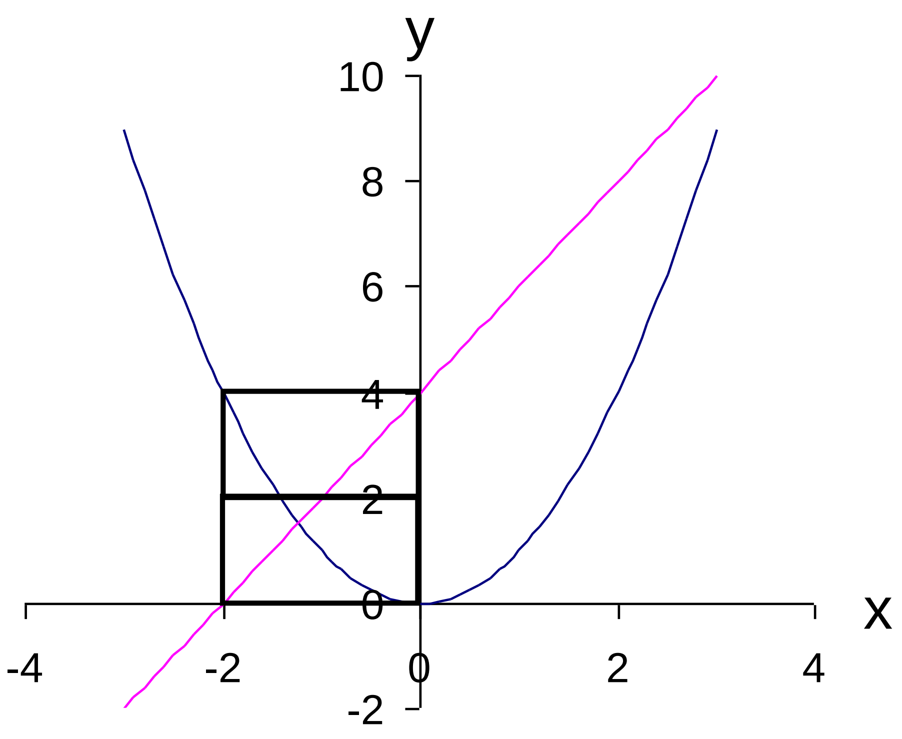


Split box

[-2,0]×[0,2]

[-2,0]×[2,4]

Example:

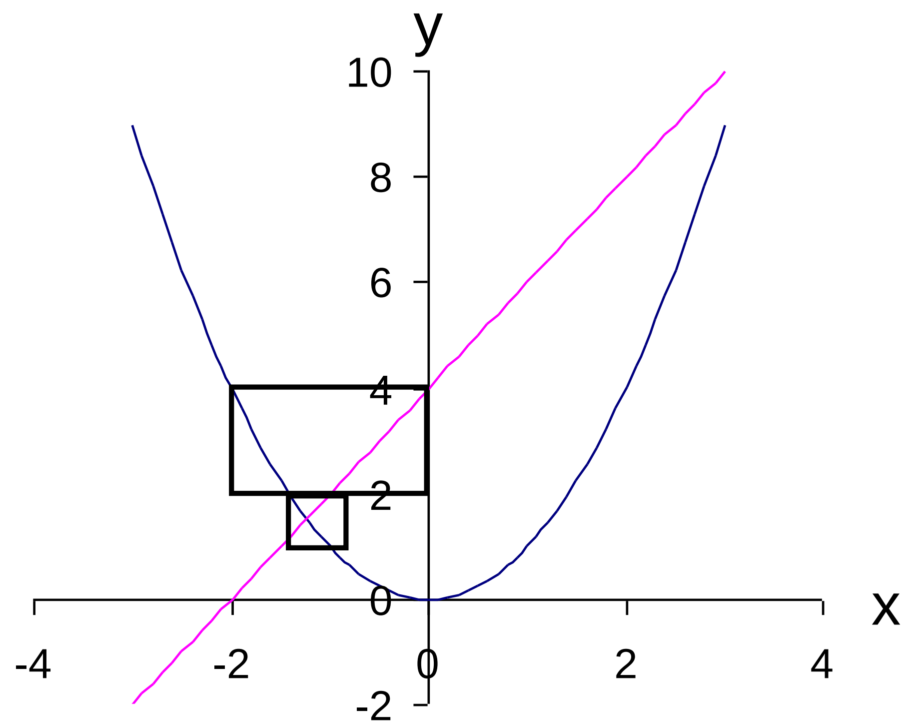
Variables: x, y

Domains: [-2,2]×[-2,10]

Constraints:

$y = x^2$

$y \geq 2x + 4$



Split box

[-2,0]×[0,2] —prune→ [-1.415,-1.082]×[1.171,2.000] (fixed point)

[-2,0]×[2,4]

Example:

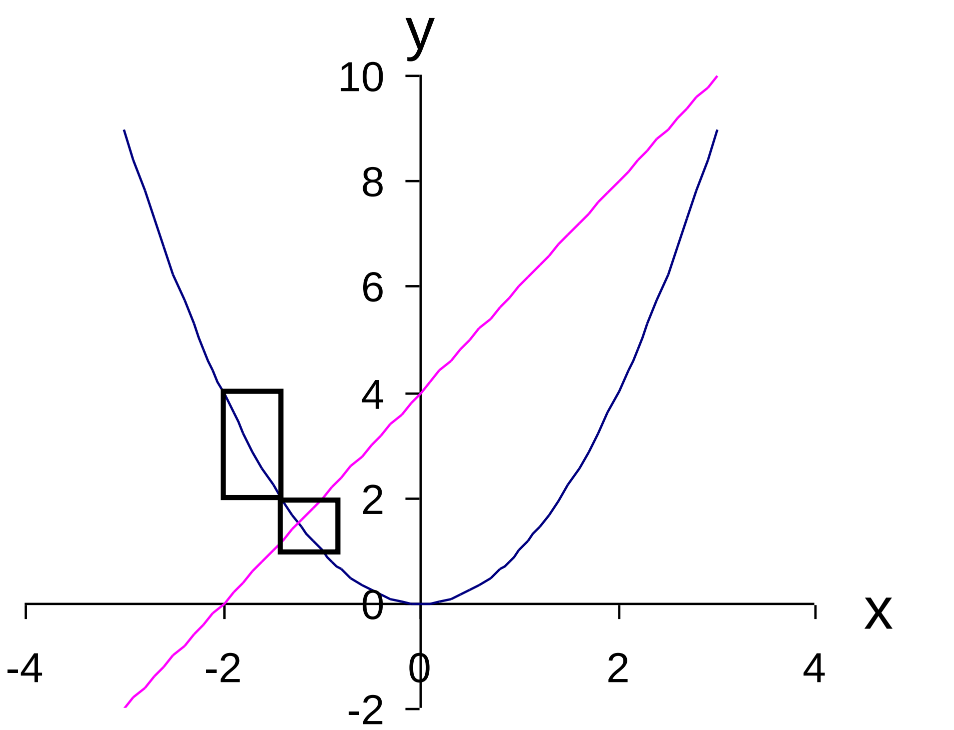
Variables: x, y

Domains: [-2,2]×[-2,10]

Constraints:

$y = x^2$

$y \geq 2x + 4$



Split box

[-2,0]×[0,2] → prune → [-1.415,-1.082]×[1.171,2.000] (fixed point)

[-2,0]×[2,4] → prune → [-2.000,-1.414]×[2.000,4.000] (fixed point)

Example:

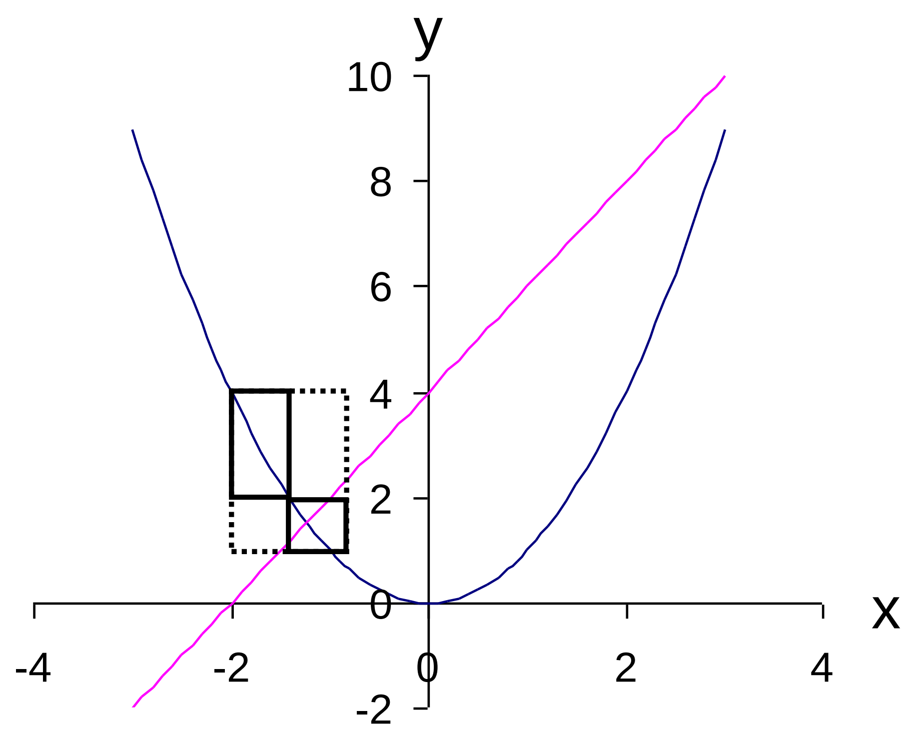
Variables: x, y

Domains: [-2,2]×[-2,10]

Constraints:

$y = x^2$

$y \geq 2x + 4$



When to stop? → Consistency requirement

if we stop now:

$[-1.415,-1.082]\times[1.171,2] \overset{+}{\cup} [-2,-1.414]\times[2,4] = [-2,-1.082]\times[1.171,4]$

Example:

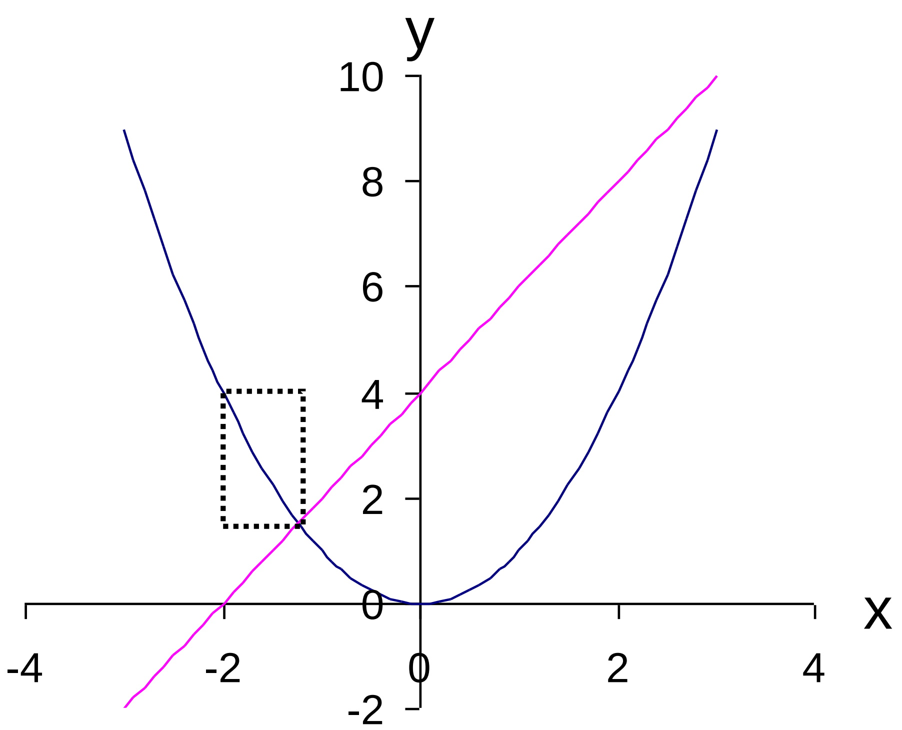
Variables: x, y

Domains: [-2,2]×[-2,10]

Constraints:

$y = x^2$

$y \geq 2x + 4$



When to stop? → Consistency requirement

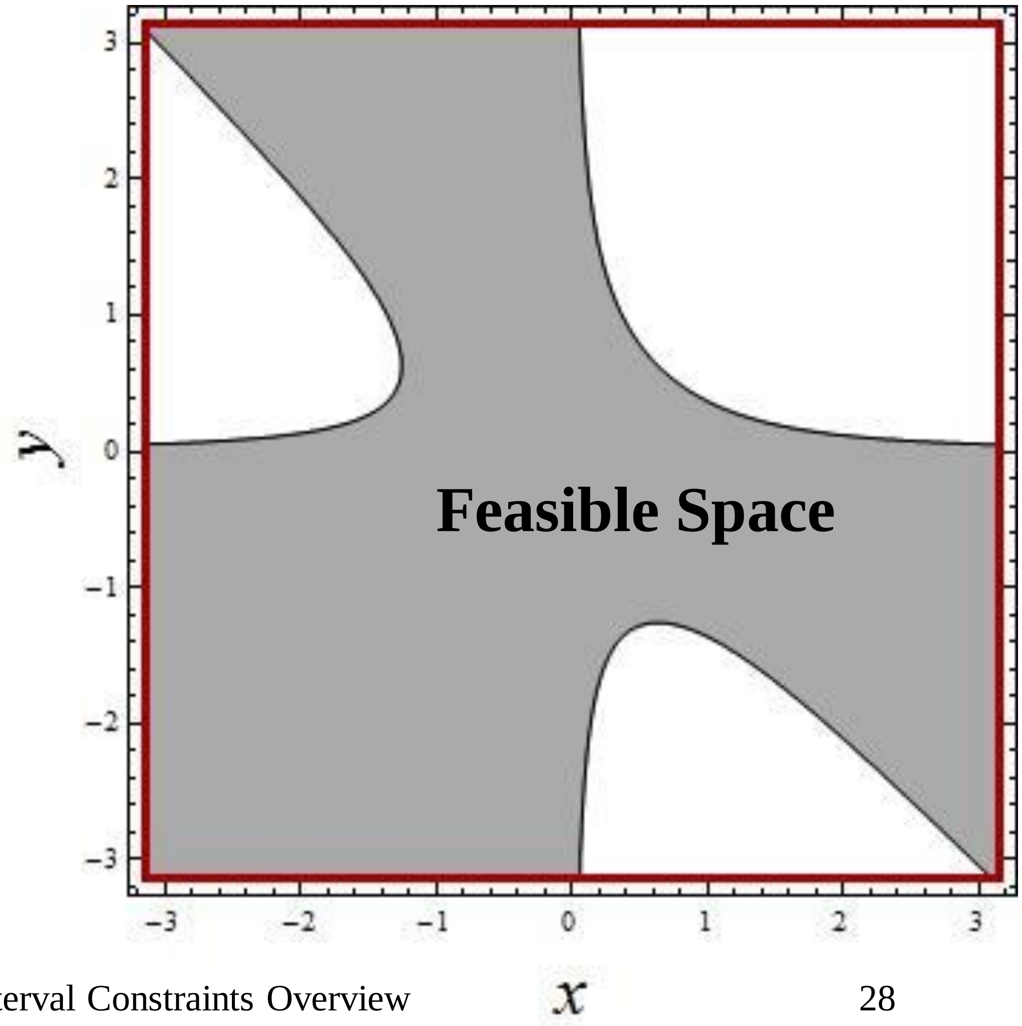
smallest box containing all canonical solutions

Continuous Constraint Programming

Continuous Constraint Satisfaction Problem:

$$x \in [-\pi, \pi] \quad y \in [-\pi, \pi]$$

$$x^2 y + x y^2 \leq 0.5$$



Continuous Constraint Programming

Continuous Constraint Satisfaction Problem:

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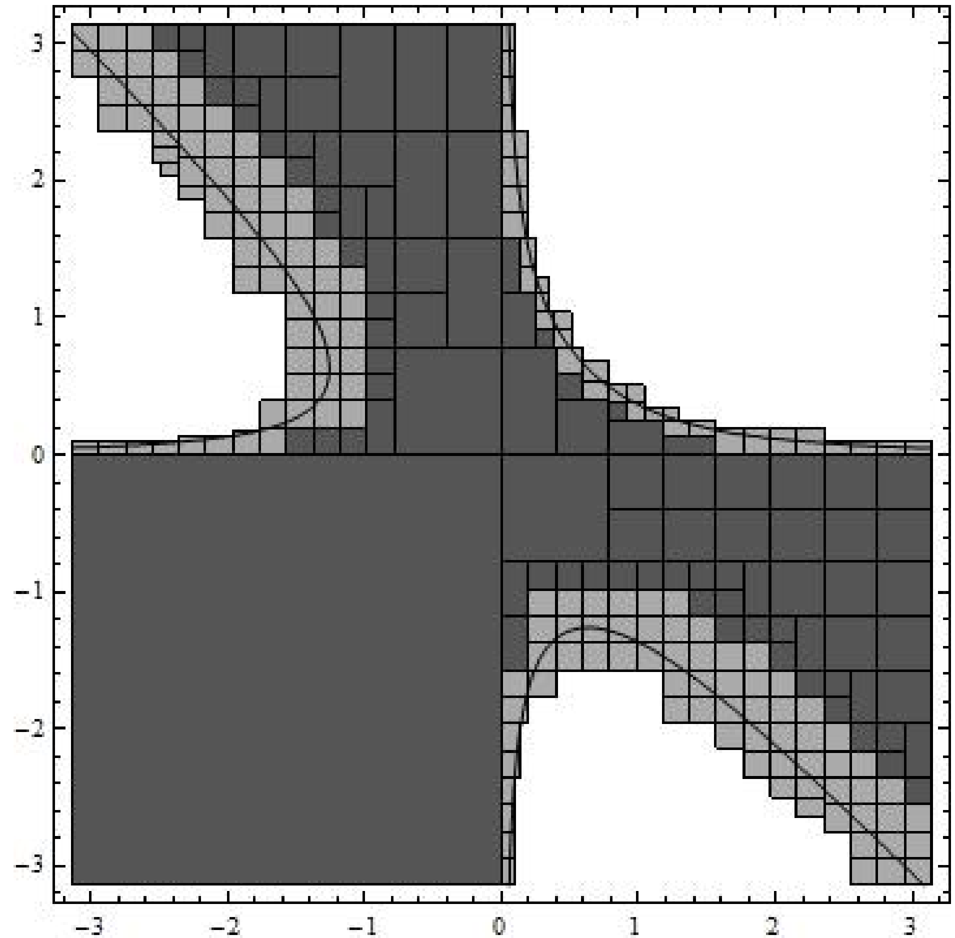


Branch & Prune Algorithms:

- **one** solution
- feasible space **box cover**

Prune Techniques:

- Interval Analysis
- Constraint Propagation

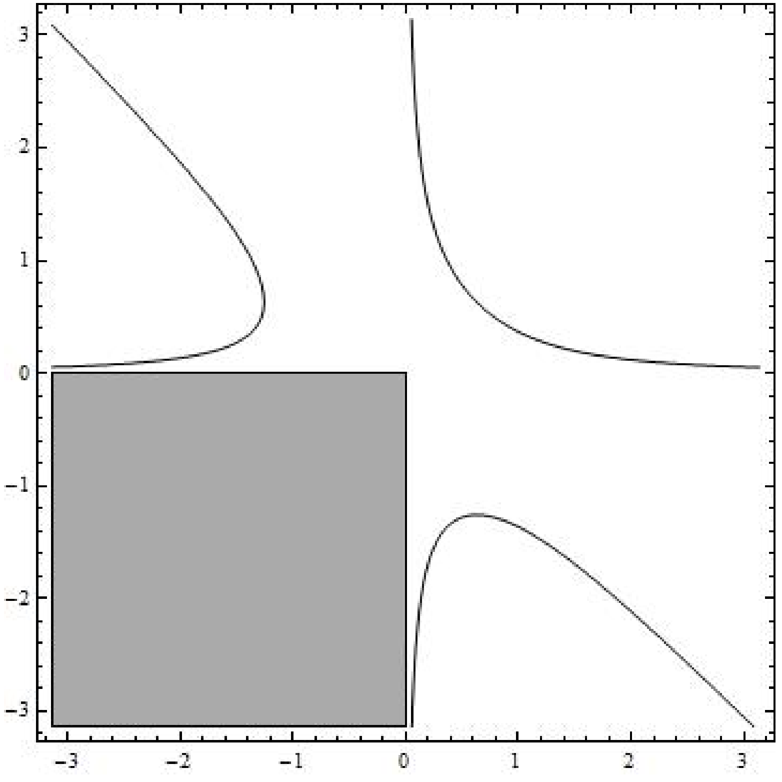
Interval Analysis

$$x^2 y + xy^2 \le 0.5$$

Inner Box ?

$x \in [-\pi, 0]$

$y \in [-\pi, 0]$



Interval Analysis

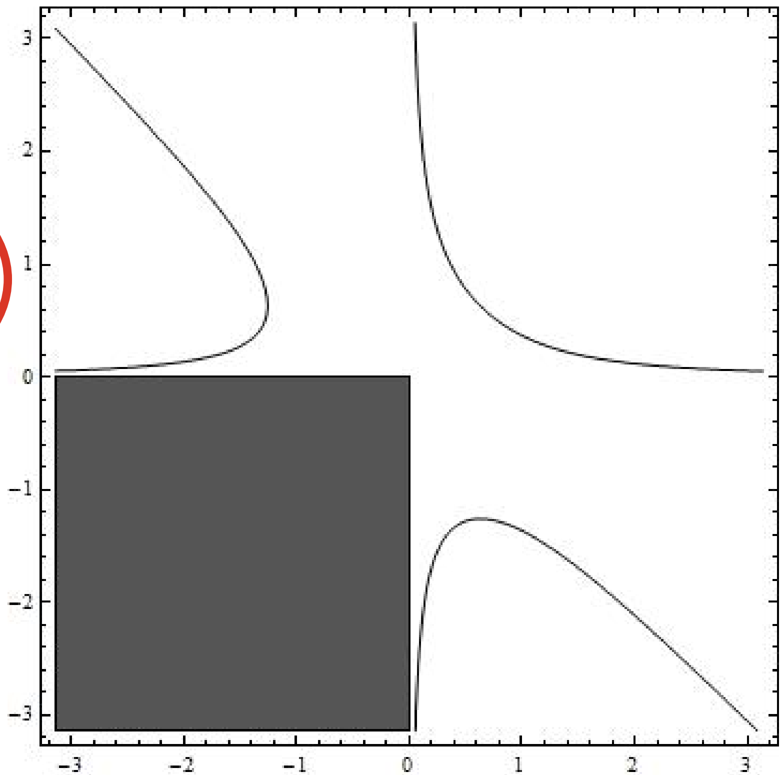
$$x^2 y + xy^2 \le 0.5$$

Interval Arithmetic

Inner Box

$$x \in [-\pi, 0]$$
$$y \in [-\pi, 0]$$

$$[-\pi,0]^2[-\pi,0] + [-\pi,0][-\pi,0]^2 =$$
$$[-2\pi^3, 0] \le 0.5$$



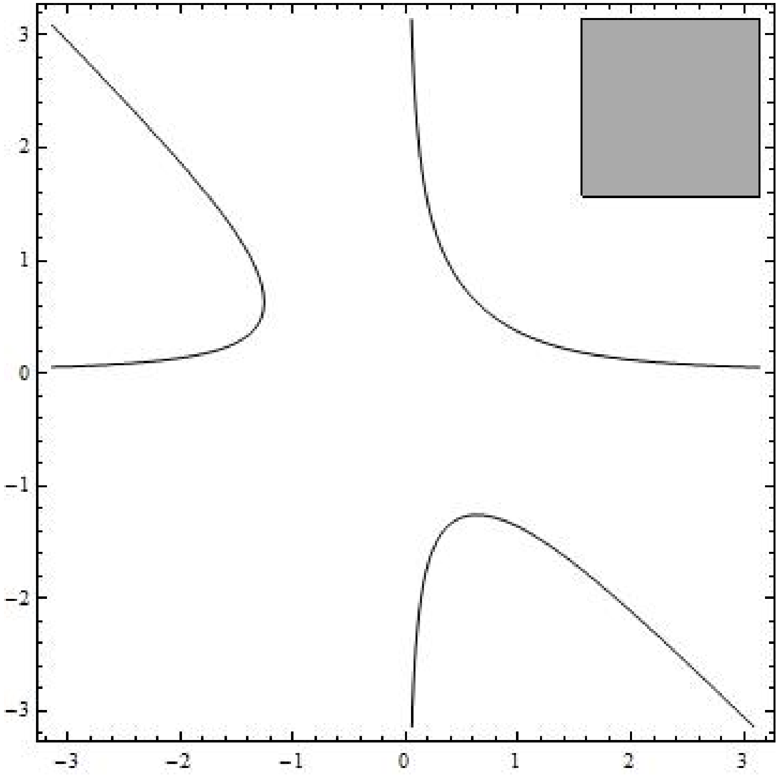
Interval Analysis

$$x^2 y + xy^2 \leq 0.5$$

Non-solution Box ?

$$x \in [\pi/2, \pi]$$

$$y \in [\pi/2, \pi]$$



Interval Analysis

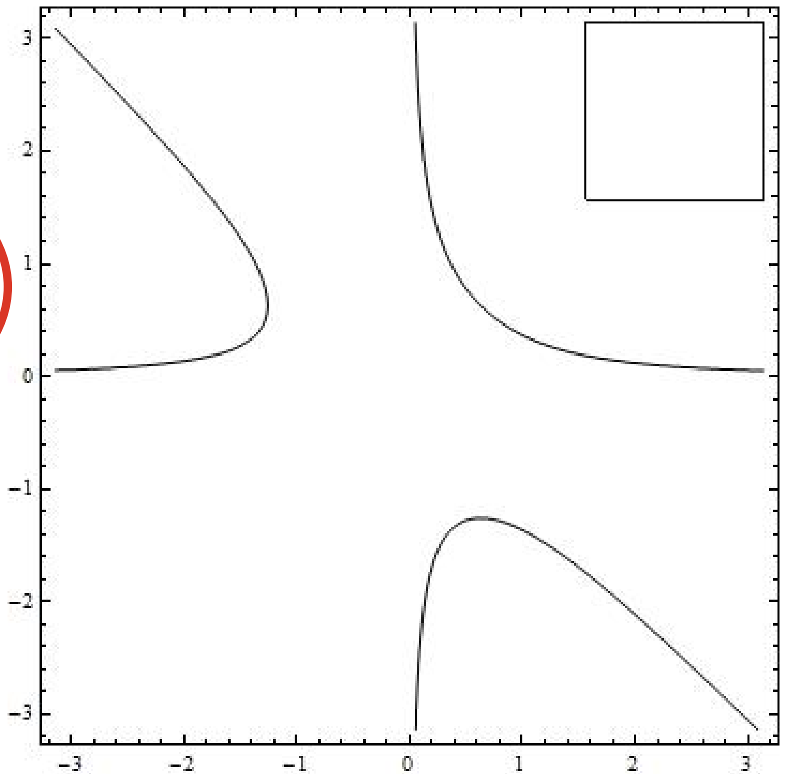
$$x^2 y + x y^2 \le 0.5$$

Interval Arithmetic

Non-solution Box

$$x \in [\pi/2, \pi]$$
$$y \in [\pi/2, \pi]$$

$$[\pi/2,\pi]^2[\pi/2,\pi] + [\pi/2,\pi][\pi/2,\pi]^2 =$$
$$[\pi^3/4, \pi^3] \le 0.5$$

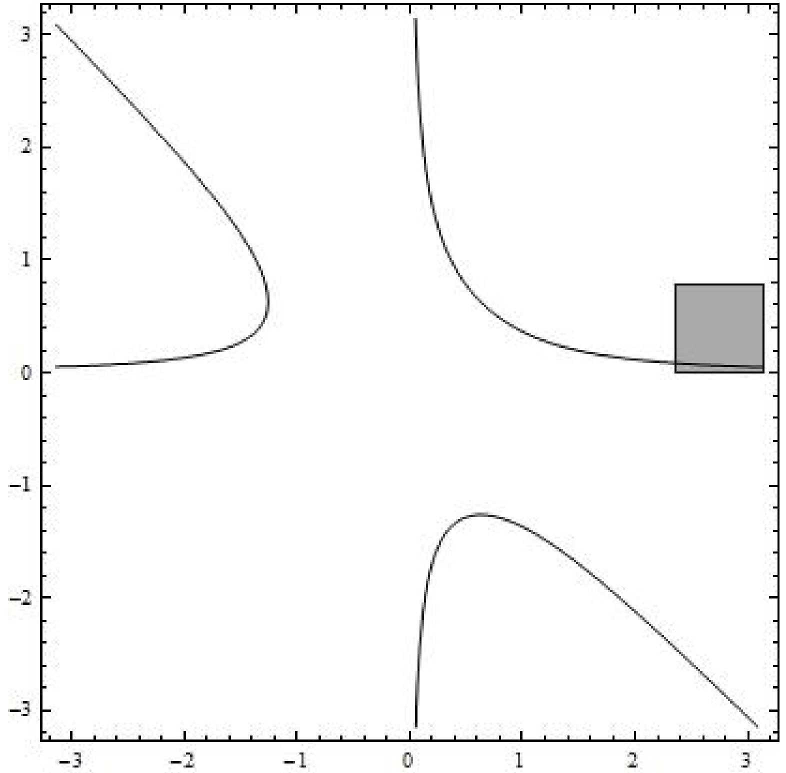


Interval Analysis

$$x^2 y + xy^2 - 0.5 \leq 0$$

Prune Boundary Box

$$[3\pi/4, \pi] \times [0, \pi/4]$$



Interval Analysis

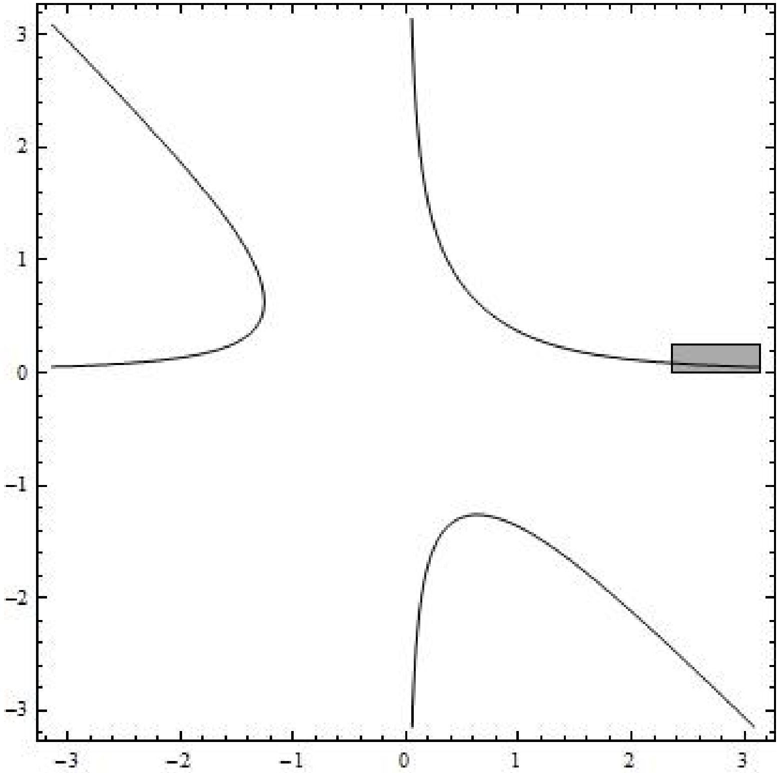
$$x^2 y + x y^2 - 0.5 \le 0$$

Prune Boundary Box

$$[3\pi/4, \pi] \times [0, \pi/4]$$

Newton step

$$[3\pi/4, \pi] \times [0, 0.2547]$$



Interval Analysis

$$x^2 y + xy^2 - 0.5 \le 0$$

Prune Boundary Box

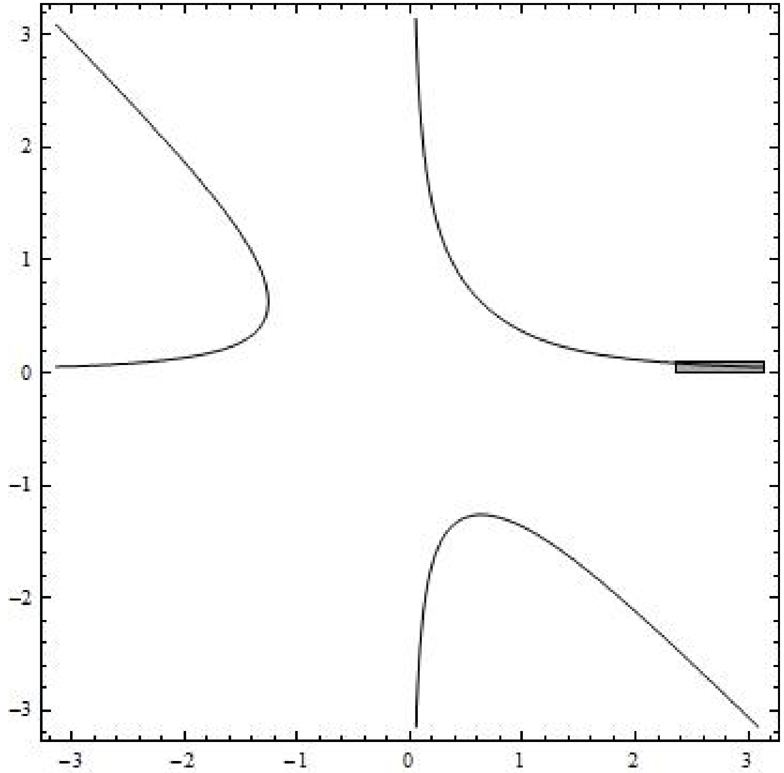
$$[3\pi/4, \pi] \times [0, \pi/4]$$

Newton step

$$[3\pi/4, \pi] \times [0, 0.2547]$$

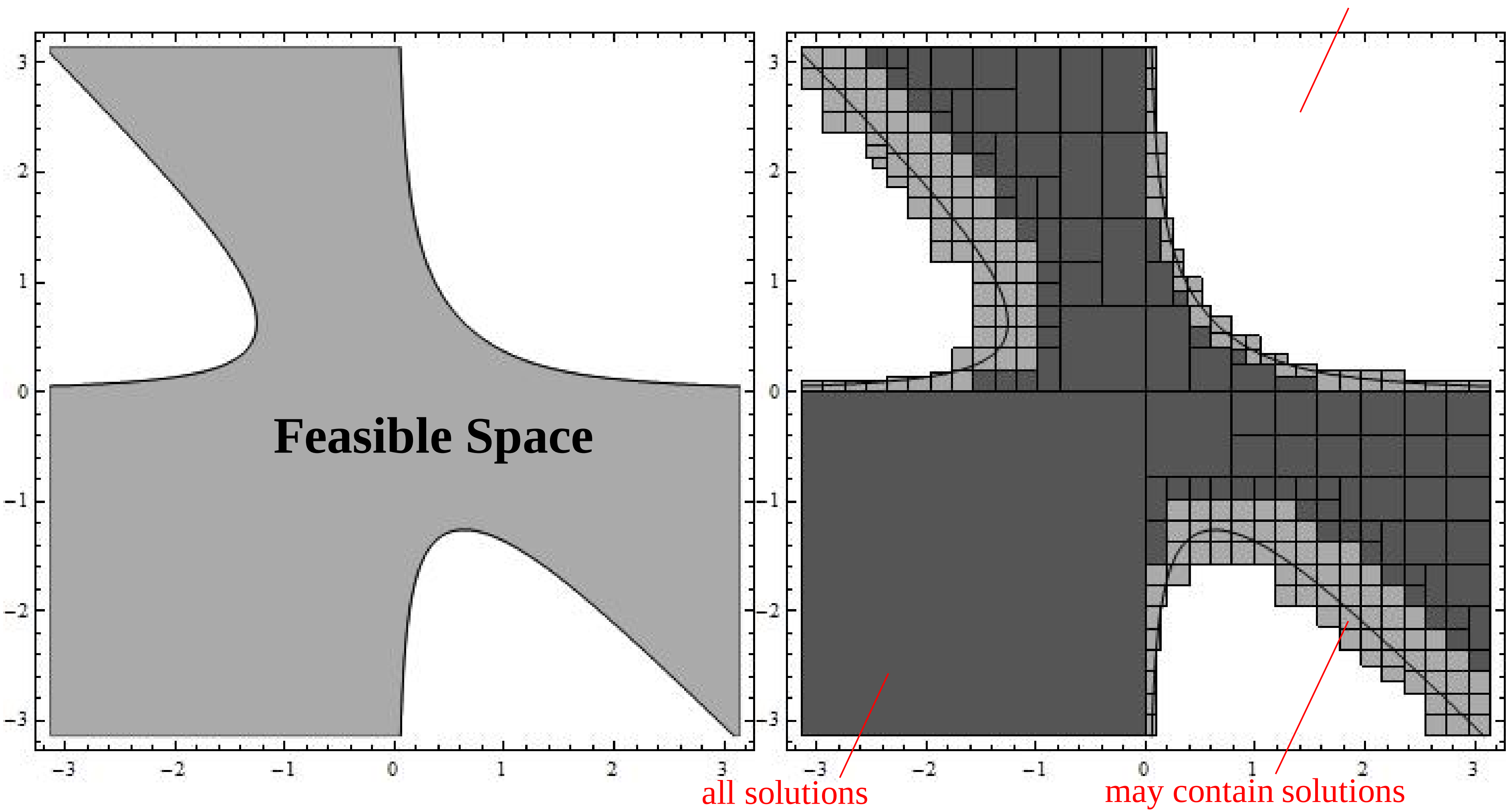
Newton steps

$$[3\pi/4, \pi] \times [0, 0.0874]$$

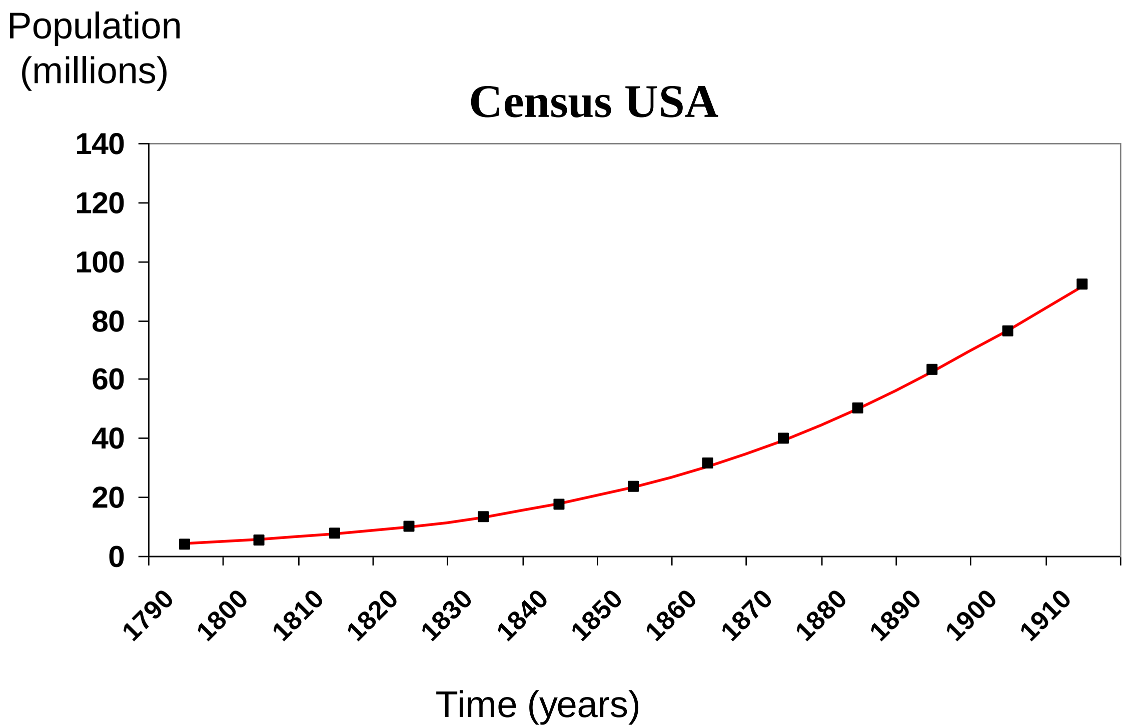


Continuous Constraint Programming

Continuous Constraint Reasoning:



A practical example:



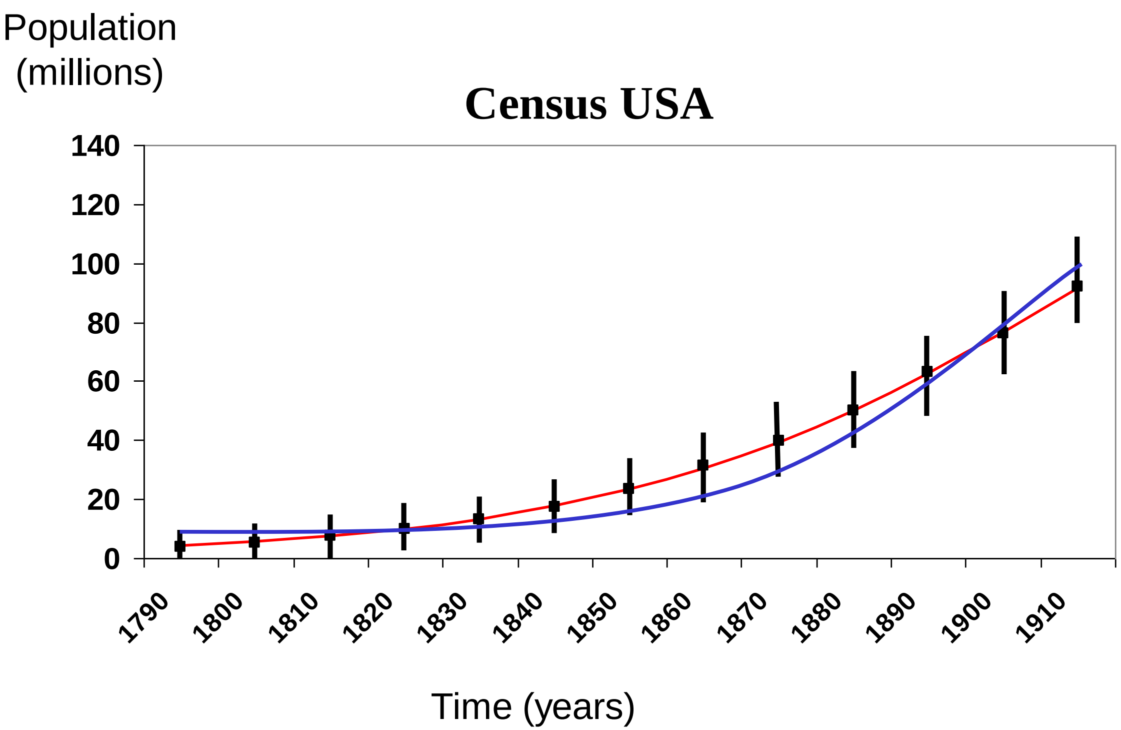
Logistic Model

$$x(t) = \frac{k x_0 e^{r(t-t_0)}}{x_0\left(e^{r(t-t_0)} - 1\right) + k}$$

result: $\begin{cases} x_0 = a \\ k = b \\ r = c \end{cases}$

Optimization Problem: $min \sum_i (x_i - v_i)^2$ with $x_i = \frac{k x_0 e^{r(t_i-t_0)}}{x_0\left(e^{r(t_i-t_0)} - 1\right) + k}$

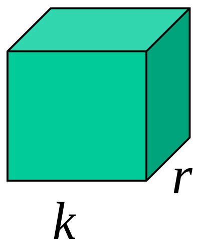
A practical example:



Logistic Model

$$x(t) = \frac{k x_0 e^{r(t-t_0)}}{x_0\left(e^{r(t-t_0)} - 1\right) + k}$$

result: x_0



CCSP: $\left\{ \langle x_0, k, r \rangle \mid \forall_{(t_i, v_i)}\; x_i = \frac{k x_0 e^{r(t_i - t_0)}}{x_0\left(e^{r(t_i - t_0)} - 1\right) + k} \;\wedge\; |x_i - v_i| \le \varepsilon_i \right\}$

Course Structure: Constraints on Continuous Domains

Lecture 1: Interval Constraints Overview

Lecture 2: Intervals, Interval Arithmetic and Interval Functions

Lecture 3: Interval Newton Method

Lecture 4: Associating Narrowing Functions to Constraints

Lecture 5: Constraint Propagation and Consistency Enforcement

Lecture 6: Problem Solving

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Important Links

- *Interval Computations*
  A primary entry point to items concerning interval computations.

- *COCONUT - COntinuous CONstraints Updating the Technology*
  Project to integrate techniques from mathematical programming, constraint programming, and interval analysis.

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