Interval Newton Method

Jorge Cruz
DI/FCT/UNL
Nov 2017

Interval Newton Method

Newton Method for Finding Roots of Univariate Functions

Interval Extension of the Newton Method

- Newton Function, Newton Step and Newton Narrowing
- Extended Interval Arithmetic for the Interval Newton Method
- Example of the Interval Newton Method

Properties of the Interval Newton Method

- Soundness
- Proving the Existence of a Solution
- Convergence and Efficiency

Enclosing the Zeros of a Family of Functions

Newton Method for Finding Roots of Univariate Functions

Let f be a real function, continuous in $[a,b]$ and differentiable in $(a..b)$

Accordingly to the mean value theorem:

$$\forall_{r_1,r_2\in[a,b]} \exists_{\xi\in[min(r_1,r_2),max(r_1,r_2)]} f(r_1)=f(r_2)+(r_1-r_2)\times f'(\xi)$$

If r_2 is a root of f then $f(r_2)=0$ and so:

$$\forall_{r_1,r_2\in[a,b]} \exists_{\xi\in[min(r_1,r_2),max(r_1,r_2)]} f(r_1)=(r_1-r_2)\times f'(\xi)$$

And solving it in order to r_2:

$$\forall_{r_1,r_2\in[a,b]} \exists_{\xi\in[min(r_1,r_2),max(r_1,r_2)]} r_2= r_1-f(r_1)/f'(\xi)$$

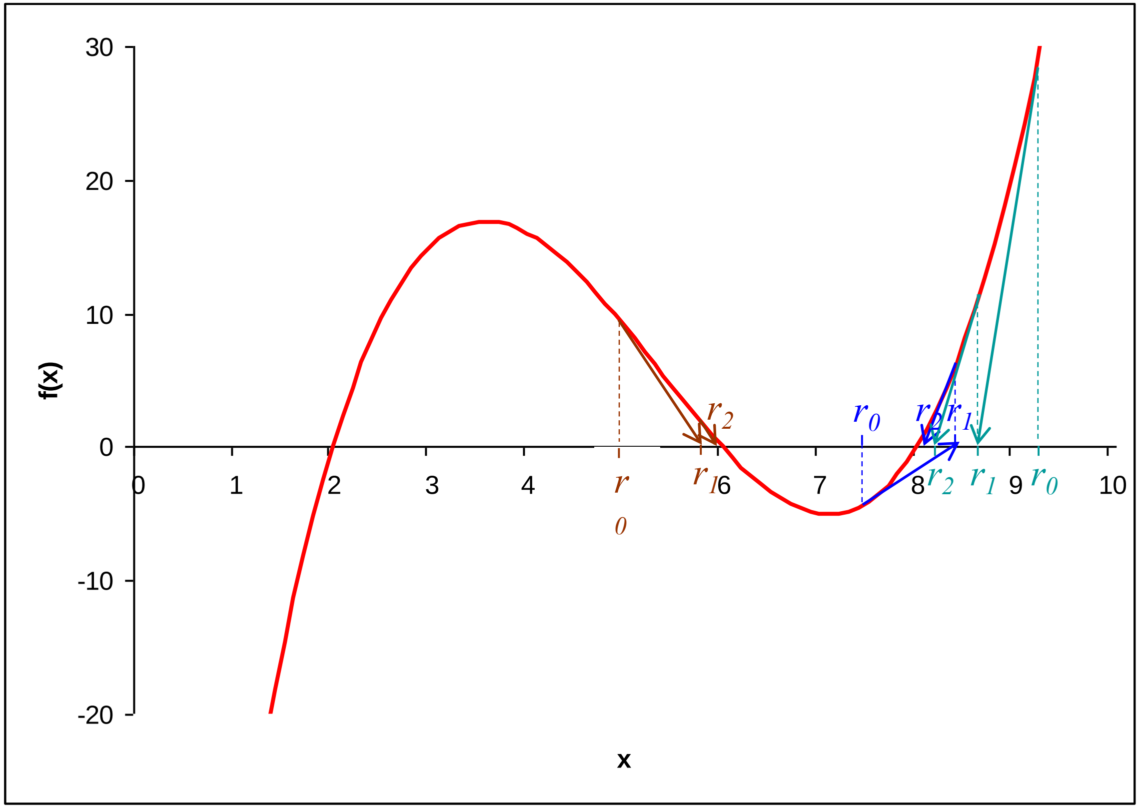
Therefore, if there is a root of f in $[a,b]$ then, from any point r_1 in $[a,b]$ the root could be computed if we knew the value of ξ

Newton Method for Finding Roots of Univariate Functions

The idea of the classical Newton method is to start with an initial value r_0 and compute a sequence of points r_i that converge to a root

To obtain r_{i+1} from r_i the value of ξ is approximated by r_i:

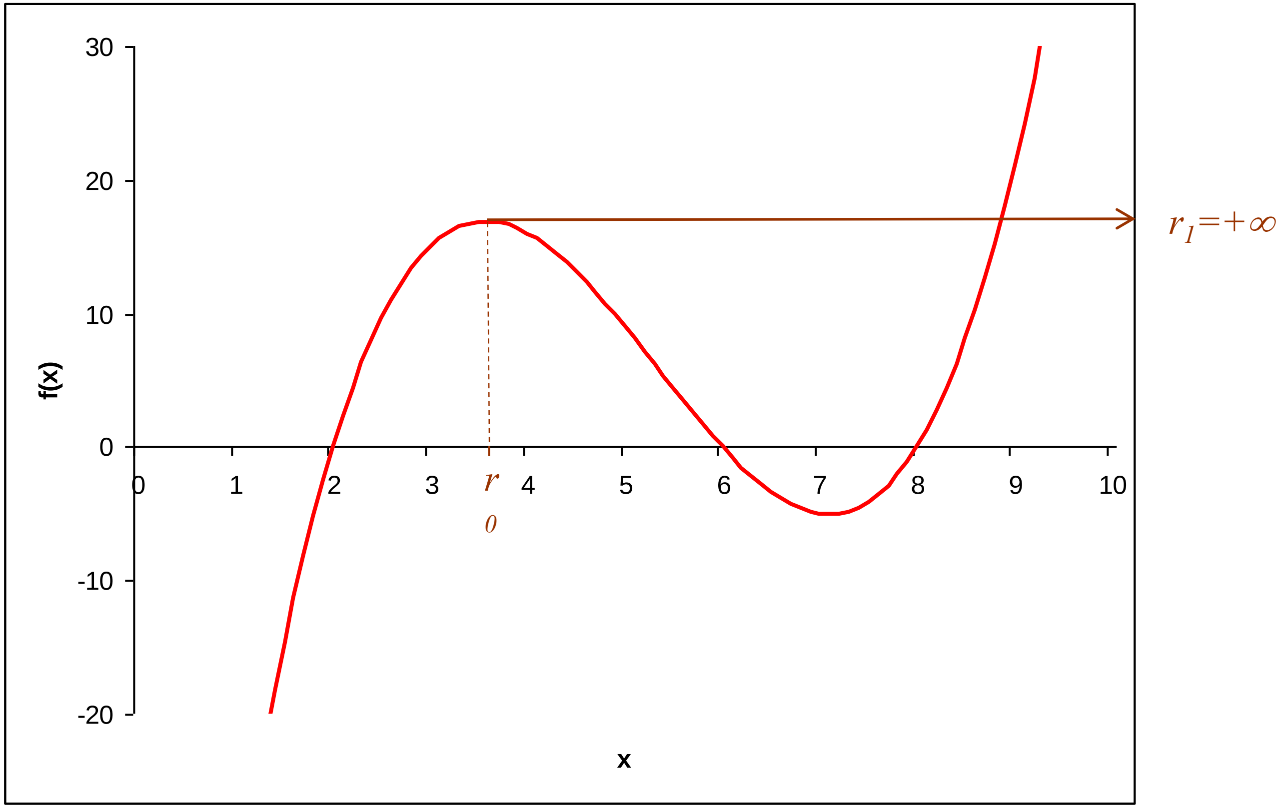
$$r_{i+1} = r_i - f(r_i)/f'(\xi) \approx r_i - f(r_i)/f'(r_i)$$



Newton Method for Finding Roots of Univariate Functions

Near roots the classical Newton method has quadratic convergence

However, the classical Newton method may not converge to a root!



Interval Extension of the Newton Method

The idea of the Interval Newton method is to start with an initial interval I_0 and compute an enclosure of all the r that may be roots

$$\forall_{r_1, r \in [a,b]} \exists_{\xi \in [min(r_1,r), max(r_1,r)]} \; r = r_1 - f(r_1)/f'(\xi)$$

If r is a root within I_0 then:

$\xi \in I_0$

$$\forall_{r_1 \in I_0} \; r \in r_1 - f(r_1)/f'(I_0)$$

(all the possible values of ξ are considered)

In particular, with $r_1 = c = center(I_0)$ we get the Newton interval function:

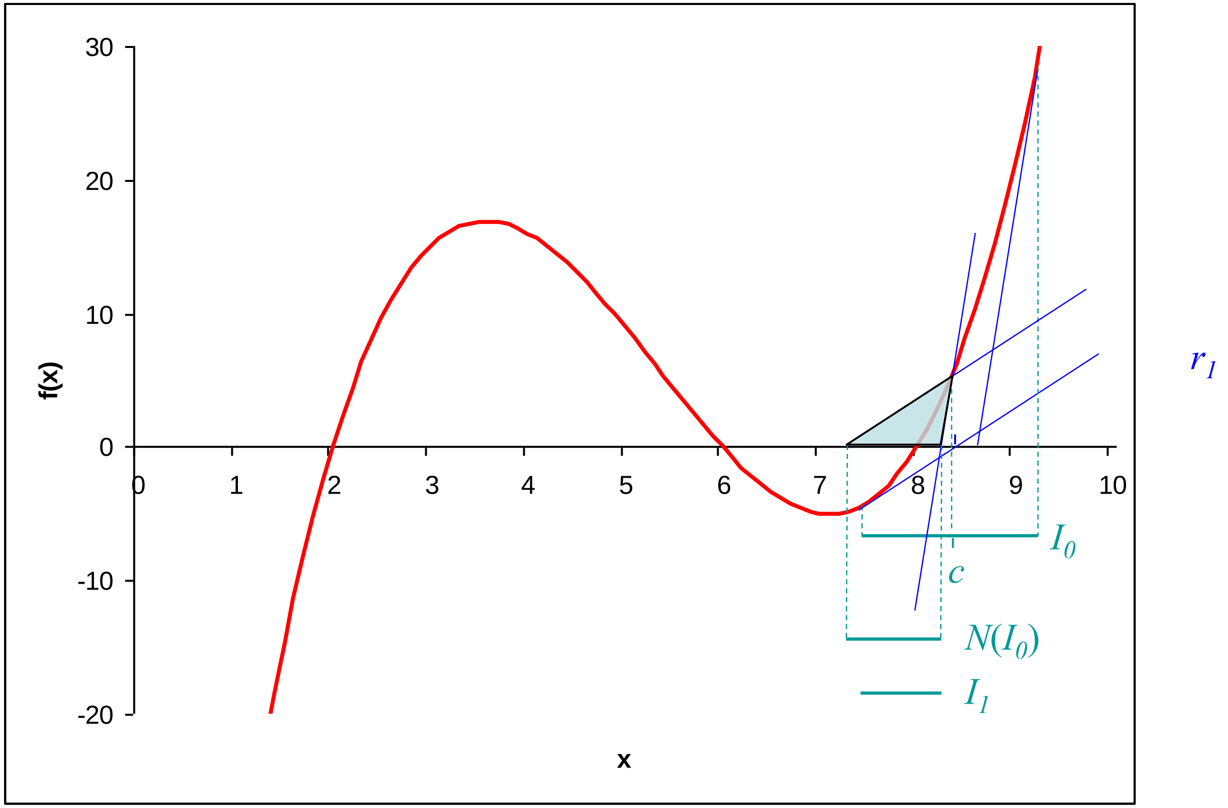
$$r \in c - f(c)/f'(I_0) = N(I_0)$$

Since root r must be within the original interval I_0, a smaller safe enclosure I_1 may be computed by:

$$I_1 = I_0 \cap N(I_0)$$

Interval Extension of the Newton Method

The idea of the Interval Newton method is to start with an initial interval I_0 and compute an enclosure of all the r that may be roots



Interval Extension of the Newton Method

Newton Function, Newton Step and Newton Narrowing

Newton Function. Let f be a real function, continuous and differentiable in the closed real interval I, and f' its derivative. Let F and F' be interval extensions of f and f', respectively. Let c be the mid value of the interval I ($c=center(I)$). The interval Newton function N with respect to f is: $N(I) = [c] - \frac{F([c])}{F'(I)}$ ❑

Newton Step. Let f be a real function, continuous and differentiable in the closed real interval I. Let N be the Newton function with respect to f. The Newton step function NS with respect to f is:

$$NS(I) = I \cap N(I)$$ ❑

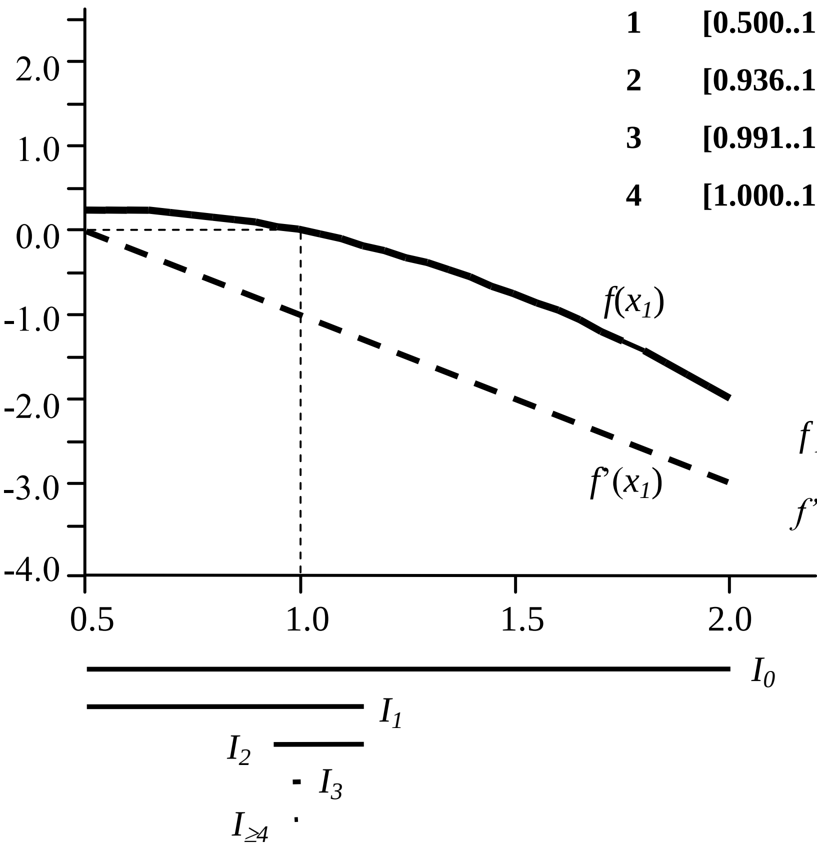
Newton Narrowing. Let f be a real function, continuous and differentiable in the closed real interval I. Let NS be the Newton step function with respect to f. The Newton narrowing function NN with respect to f is:

$$NN(I) = \begin{cases} \varnothing & \text{if} \quad NS(I)=\varnothing \\ I & \text{if} \quad NS(I)=I \\ NN(NS(I)) & \text{if} \quad NS(I)\subset I \end{cases}$$ ❑

Interval Extension of the Newton Method

Example of the Interval Newton Method

i	I_i	$[c_I]$	$F_E([c_I])$	$F'_E(I_i)$	$N(I_i)$
0	[0.500..2.000]	{1.250}	[-0.313..-0.312]	[-3.000..0.000]	[-∞..1.146]
1	[0.500..1.146]	{0.823}	[0.145..0.146]	[-1.292..0.000]	[0.936..+∞]
2	[0.936..1.146]	{1.041}	[-0.042..-0.042]	[-1.292..-0.872]	[0.991..1.009]
3	[0.991..1.009]	{1.000}	[0.000..0.000]	[-1.018..-0.982]	[1.000..1.000]
4	[1.000..1.000]	{1.000}	[0.000..0.000]	[-1.000..-1.000]	[1.000..1.000]



$f_E \equiv x_1 - x_1^2$ $\qquad$ $F_E \equiv X_1 - X_1^2$ $\qquad$ $[c_I] = [center(I_i)]$

$f'_E \equiv 1 - 2x_1$ $\qquad$ $F'_E \equiv 1 - 2X_1$

Interval Extension of the Newton Method

Extended Interval Arithmetic for the interval Newton Method

Using extended interval arithmetic, the result of the Newton function is not guaranteed to be a single interval:

> division by an interval containing zero may yield the union of two intervals

The solution could be to use the union hull of the obtained intervals

A much better approach is to intersect separately each obtained interval with the original interval and then:

> If the result of the intersection is a single interval, the Newton narrowing can normally continue.
> Otherwise, the union hull of the obtained intervals should be considered

Analysis of the Newton Steps

Before a more detailed analysis of the Newton method we remind the division of two finite intervals , **I** and **J** where **I = [a , b]** and **J = [c , d]**

Case 1. $0 \notin I$ & $0 \notin J$ i.e. (a>0 | b<0) & (c>0 | d<0)

I/J : one finite interval not containing 0

I / J = [min(a/d , b/c) , max(b/d , a/c)]

Case 2. $0 \notin I$ & $0 \in J$ i.e. (a>0 | b<0) & (c < 0 < d)

I/J : two semi-infinite intervals not including 0

I/J = [-∞, min(a/c , b/d)] ∪ [max(a/d , b/c) , +∞]

Case 3. $0 \in I$ & $0 \notin J$ i.e. (a < 0 < b) & (c>0 | d<0)

I/J: one finite interval containing 0

I/J = [min(a/c , b/d) , max(b/c , a/d)]

Case 4. $0 \in I$ & $0 \in J$ i.e. (a < 0 < b) & (c < 0 < d)

I/J: one infinite interval (degenerate)

I/J = [-∞, +∞]

Analysis of the Newton Steps

Newton Step: $I_{n+1} = I_n \cap N(I_n)$

where $N(I_n) = m_n - X_n$ is the (centered) Newton Function

i.e. $m_n = mid(I_n)$ and $X = F(m_n) / F'(I_n)$

Also let $F(m_n) = [a_n, b_n]$ and $F'(I_n) = [c_n, d_n]$

Case 1. $0 \notin F(m_n)$ & $0 \notin F'(I_n)$:

$X_n = [\min(a_n/d_n, b_n/c_n) , \max(b_n/d_n, a_n/c_n)]$

X_n is one finite interval not containing 0

$N(I_n) = m_n - X_n$ is one finite interval not containing m_n

$I_{n+1} = I_n \cap N(I_n) \subseteq I_n$

$m_n \in I_n$ and $m_n \notin N(I_n)$

Moreover, since m_n is the mid-point of I_n

$$width(I_{n+1}) < 0.5\ width(I_n)$$

The Newton step yields one interval with, at most, **half the width of I_n**.
If I_n contains a zero of the function then, I_{n+1} also does contain this zero.
Hence, if I_{n+1} is empty, there were no zeros of **F** (nor f) in I_n (**no zeros are lost**).
However, in this case and *due to evaluation errors*, I_{n+1} **might be not empty.**

Analysis of the Newton Steps

Newton Step: $I_{n+1} = I_n \cap N(I_n)$

where $N(I_n) = m_n - X_n$ is the (centered) Newton Function

i.e. $m_n = mid(I_n)$ and $X = F(m_n) \,/\, F'(I_n)$

Also let $F(m_n) = [a_n, b_n]$ and $F'(I_n) = [c_n, d_n]$

Case 2. $0 \notin F(m_n)$ & $0 \in F'(I_n)$:

$X_n = I^1_n \cup I^2_n = [-\infty, \min(a_n/c_n, b_n/d_n)] \cup [\max(a_n/d_n, b_n/c_n), +\infty]$

X_n **is composed of two semi-infinite intervals not including 0**

$N(I_n) = m_n - I^1_n \cup m_n - I^2_n$ are two semi-infinite intervals not containing m_n

$I_{n+1} = (I_n \cap m_n - I^1_n) \cup (I_n \cap m_n - I^2_n)) \subset I_n$

Now, I_{n+1} may be empty or

two finite intervals not containing m_n; or

one finite interval not containing m_n; or

The Newton step yields two intervals, each at most with **half the width of I_n.**
If an **interval is typically empty,** it does not contain a zero of the function.
Again, a non empty interval may contain no zeros of the function.

Analysis of the Newton Steps

Newton Step: $I_{n+1} = I_n \cap N(I_n)$

where $N(I_n) = m_n - X_n$ is the (centered) Newton Function

i.e. $m_n = mid(I_n)$ and $X = F(m_n) \,/\, F'(I_n)$

Also let $F(m_n) = [a_n, b_n]$ and $F'(I_n) = [c_n, d_n]$

Case 3. $0 \in F(m_n)$ & $0 \notin F'(I_n)$:

$X_n = [\min(a_n/c_n, b_n/d_n), \max(b_n/c_n, a_n/d_n)]$

X_n is one finite interval containing 0

$N(I_n) = m_n - X_n$ is one finite interval containing m_n

$I_{n+1} = I_n \cap N(I_n) \subseteq I_n$

Now, I_{n+1} may or may not be strictly included in I_n

Since $F(m_n)$ includes zero, we are already close to a zero of the function. In fact, without rounding errors, the zero would have been found!

$$F(m_n) = 0 \Rightarrow f(m_n) = 0.$$

The Newton step yields **one interval, possibly strictly included in I_n.**

If not strictly smaller, should it be split, and Newton steps applied to the splits?

Analysis of the Newton Steps

Newton Step: $I_{n+1} = I_n \cap N(I_n)$

where $N(I_n) = m_n - X_n$ is the (centered) Newton Function

i.e. $m_n = mid(I_n)$ and $X = F(m_n) / F'(I_n)$

Also let $F(m_n) = [a_n, b_n]$ and $F'(I_n) = [c_n, d_n]$

Case 4. $0 \in F(m_n)$ & $0 \in F'(I_n)$:

$X_n = [-\infty, +\infty]$

X_n is one infinite interval

$N(I_n) = m_n - X_n$ is one infinite interval

$I_{n+1} = I_n \cap N(I_n) = I_n$

The Newton step reaches a fixed point, i.e. I_{n+1} does not narrow I_n.
Again, without rounding errors, the zero would have been found!
Again, since $I_{n+1} = I_n$, should it be split, and Newton steps applied to the splits?

Stopping Criteria

In general, when an interval is not narrowed by the Newton Step (due to evaluation errors, or in cases 3 and 4) we may consider splitting and applying the Newton Step to each of the resulting subintervals.

The following criteria specify situations when we **may chose not to split** the intervals any further.

Situation 1. We are already close to a solution. Let ε_x and ε_f be arbitrarily small reals.

Criterion A: $\mathtt{width}(I_n) < \varepsilon_x$

Do not apply the Newton Step to an interval I_n if $\mathtt{width}(I_n) < \varepsilon_x$ since we already obtained a good approximation of the zero.

Criterion B: $|F(I_n)| < \varepsilon_F$

Do not apply the Newton Step to an interval I_n if $|(F(I_n)| < \varepsilon_x$ since in the considered interval the value of the function is already "sufficiently" close to zero.

Stopping Criteria

Situation 2. No further convergence due to rounding errors.

Criterion C: $0 \in F(m_n)$ & $0 \notin F'(I_n)$ and $I_{n+1} \supset I_n$

$I_{n+1} \supset I_n$ means that the Newton Step does not narrow a given interval.

But since $0 \notin F'(I_n)$ then the function is monotonic (increasing or decreasing) in the interval and it is very likely that a zero lies in this interval.

However, this is not guaranteed – the **evaluation** of $F(m_n)$ may produce a large approximation error and contain a 0 even if m_n is not 0.

Given the rounding errors, there is little we can do to narrow I_n … except

a) use a higher precision in the computations; or

b) Use a point $k_n \in I_n$ different from its midpoint $k_n \neq m_n$.

Stopping Criteria

Situation 3. Degenerate narrowing.

Criterion D: $0 \in F(m_n)$ & $0 \in F'(I_n)$ & $R > 1024$ (!?)

This case may arise either because $F(m_n)$ is very wide (due to rounding errors) or because we are already very close to a solution.

To discard the first case, we may check the effect of the rounding by comparing the widths of the evaluation of F in a single point and in the whole interval. Let us define the ratio

$$R' = width(F(I_n))/width(F(m_n))$$

If R is sufficiently large, then this is an indication that the rounding errors are not "significant" and we are already close to a solution. In fact, to avoid computing $F(I_n)$, and since $F'(I_n)$ needs to be computed, we may use a good approximation

$$R = width(F'(I_n))/width(F(m_n))$$

Properties of the Interval Newton Method

Soundness

If a zero of a function is searched within an interval then it may be searched within a possibly narrower interval obtained by the Newton narrowing function with the guarantee that no zero is lost

Soundness of the Interval Newton Method with Roots. Let f be a real function, continuous and differentiable in the closed real interval I. If there exists a zero r_0 of f in I then r_0 is also in $N(I)$, $NS(I)$ and $NN(I)$, where N, NS and NN are respectively the Newton function, the Newton step function and the Newton narrowing function with respect to f:

$$\forall_{r_0 \in I}\ f(r_0)=0 \Rightarrow r_0 \in N(I) \wedge r_0 \in NS(I) \wedge r_0 \in NN(I)$$ ❑

If the result of the Newton narrowing function is the empty set then the original interval does not contain any zero of the real function

Soundness of the Interval Newton Method without Roots. Let f be a real function, continuous and differentiable in the closed real interval I. If $NS(I)=\varnothing$ or $NN(I)=\varnothing$ (where NS and NN are respectively the Newton step function and the Newton narrowing function with respect to f) then there is no zero of f in I:

$$NS(I)=\varnothing \vee NN(I)=\varnothing \Rightarrow \neg\exists_{r_0 \in I}\ f(r_0)=0$$ ❑

Properties of the Interval Newton Method

Proving the Existence of a Solution

Despite its soundness, the method is not complete: in case of non existence of a root the result is not necessarily the empty set

Therefore obtaining a non empty set does not guarantee the existence of a root

However, in some cases, the Newton method may guarantee the existence of a root

Interval Newton Method to Prove the Existence of a Root. Let f be a real function, continuous and differentiable in the closed real interval I. Let N be the Newton function wrt f. If the result of applying the Newton function to I is included in I then there exists a zero of f in I:

$$N(I) \subseteq I \Rightarrow \exists_{r_0 \in I}\, f(r_0)=0$$ ❑

Properties of the Interval Newton Method

Convergence and Efficiency

The interval arithmetic evaluation of any Newton narrowing function is guaranteed to stop

> **Convergence of the Interval Newton Method.** Let *f* be a real function, continuous and differentiable in the closed real interval *I*. The interval arithmetic evaluation of the Newton narrowing function (*NN*) with respect to *f* will converge (to an *F*-interval or the empty set) in a finite number of Newton steps (*NS*). ❑

Convergence may be quadratic for small intervals around a simple zero of the real function:

$$width(NS^{(n+1)}(I_0)) \leq k \times (width(NS^{(n)}(I_0))^2$$

Moreover, even for large intervals the rate of convergence may be reasonably fast (geometric):

$$\text{If } 0 \notin F([c]) \text{ and } 0 \notin F'(I) \text{ then } width(NS(I)) \leq 0.5 \times width(I)$$

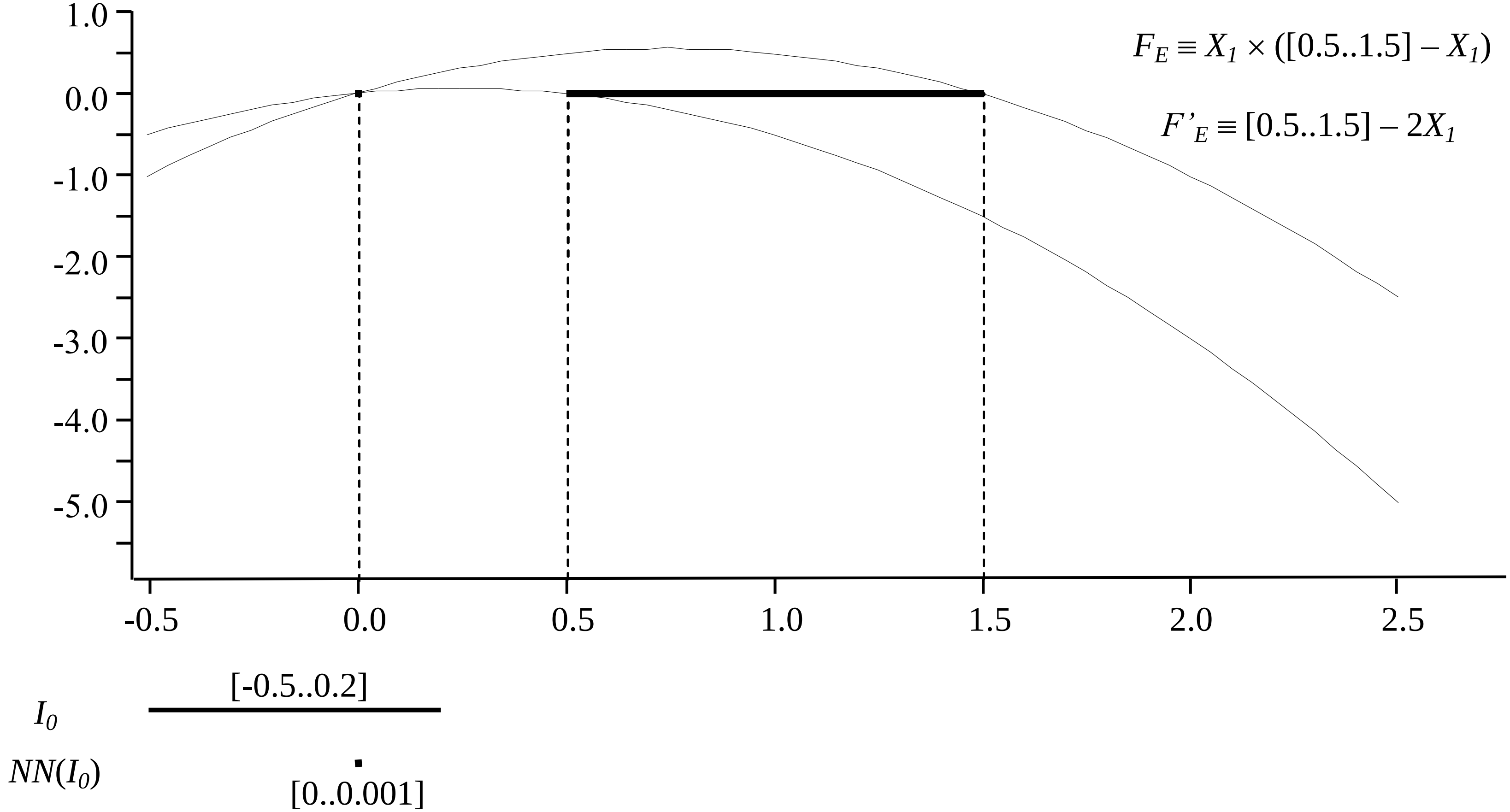
Enclosing the Zeros of a Family of Functions

The method can be naturally extended to deal as well with real functions that include parametric constants represented by intervals

The intended meaning is to represent the family of real functions defined by any possible real instantiation for the interval constants

The existence of a root means that there is a real valued combination, among the variable and all the interval constants, that zeros the function

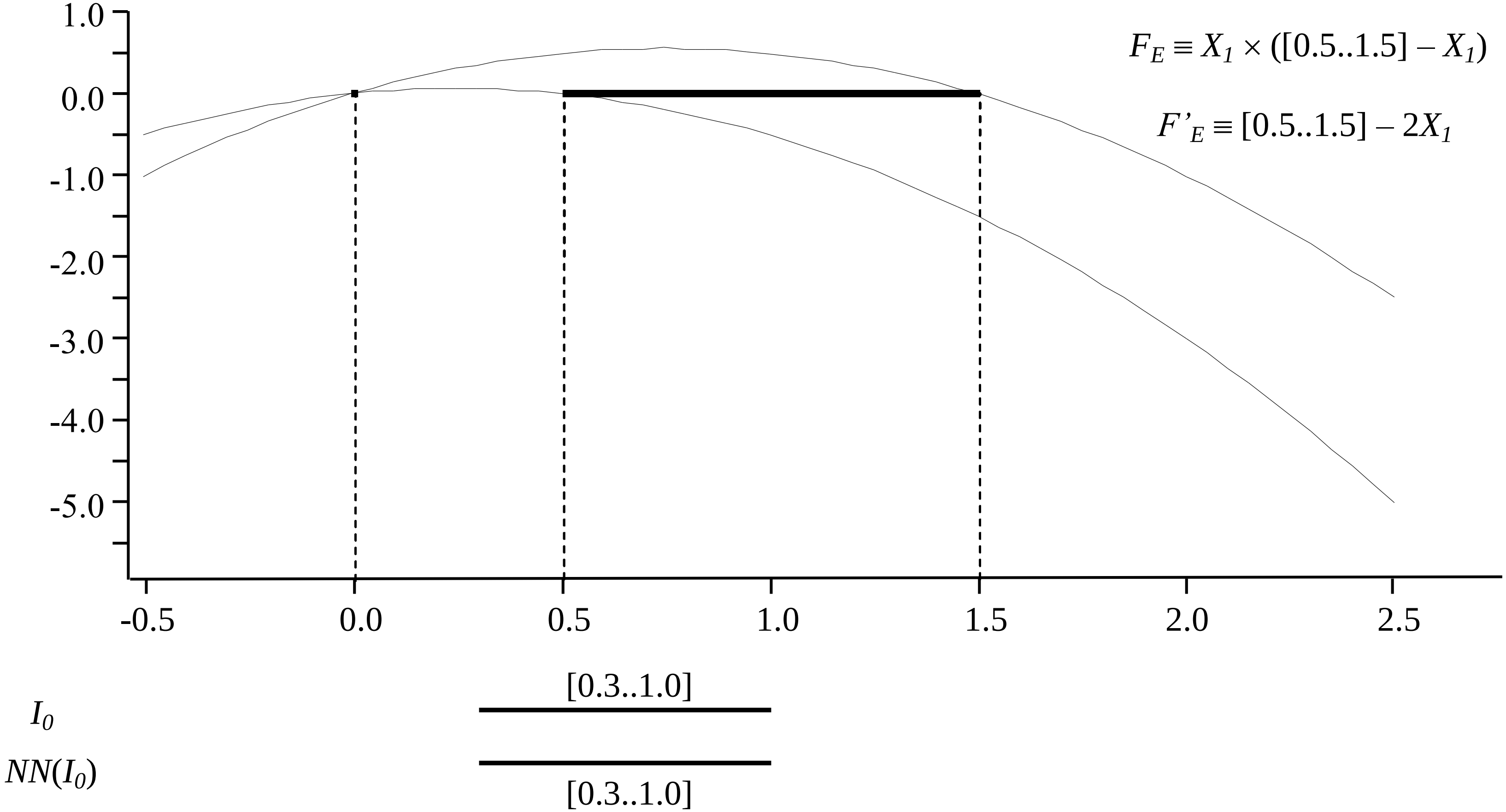
Enclosing the Zeros of a Family of Functions



If the initial interval is [-0.5,0.2] the unique zero is successfully enclosed within a canonical *F*-interval [0..0.001]

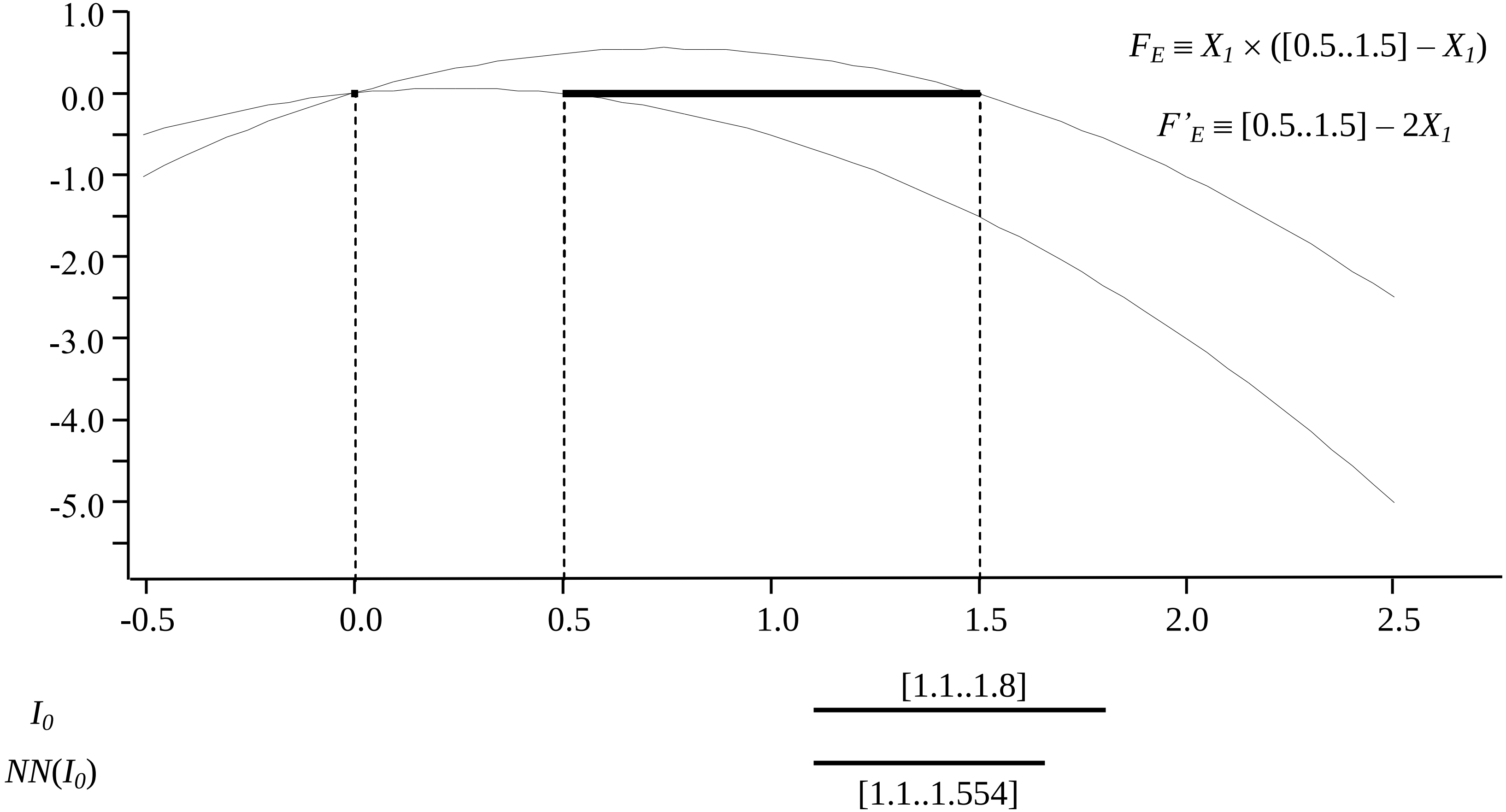
(assuming that the canonical width is 0.001)

Enclosing the Zeros of a Family of Functions



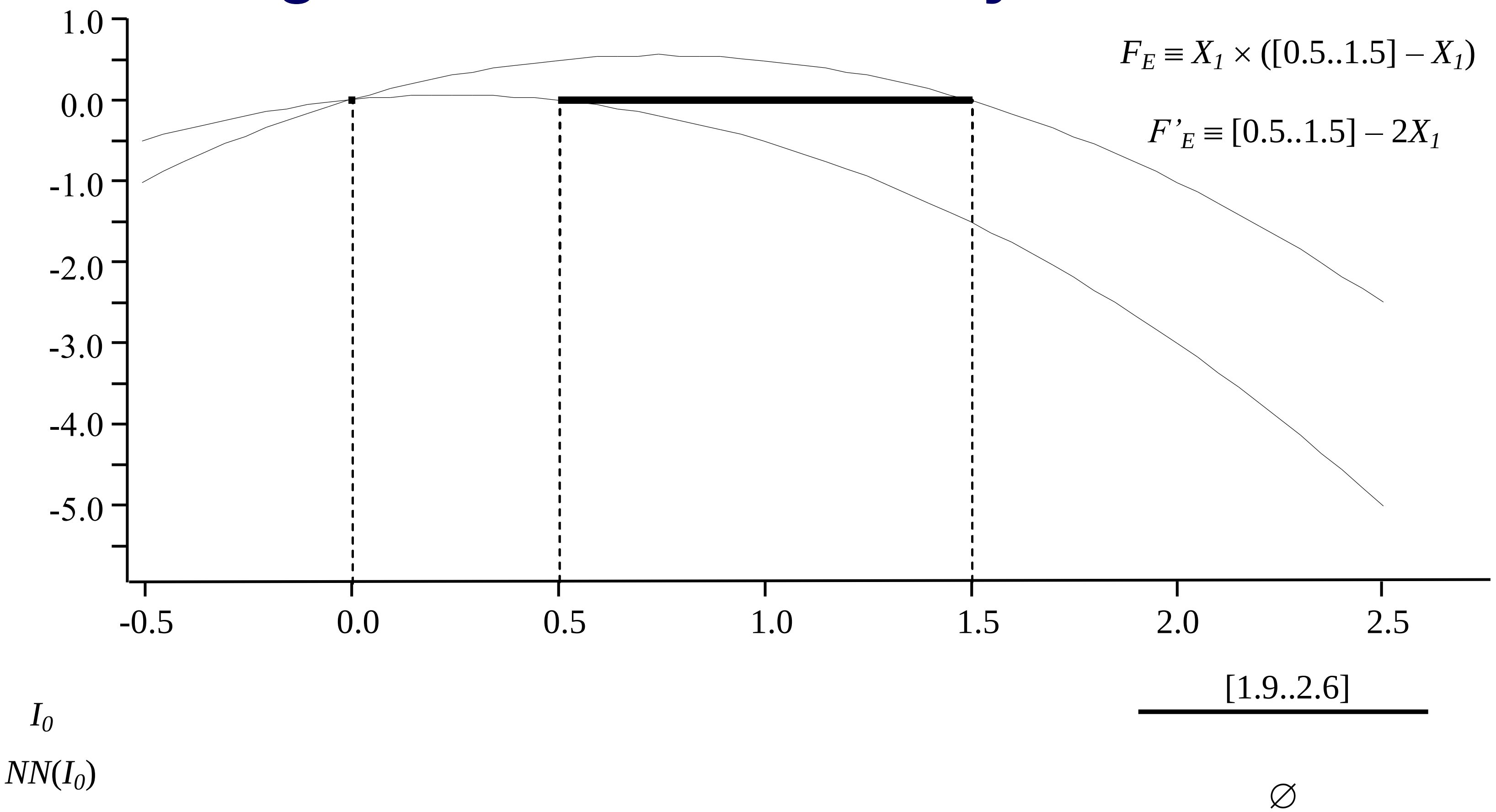
If the initial interval is [0.3,1.0] it cannot be narrowed because both $F_E([0.65])$ and $F'_E([0.3..1.0])$ include zero

Enclosing the Zeros of a Family of Functions



If the initial interval is [1.1,1.8] the right bound is updated to 1.554

Enclosing the Zeros of a Family of Functions



If the initial interval is [1.9,2.6] it can be proven that it does not contain any zeros