

# Intervals, Interval Arithmetic and Interval Functions

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# Intervals, Interval Arithmetic and Interval Functions

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# Basic Concepts

**Constraint.** A constraint  $c$  is a pair  $(s, \rho)$ , where  $s$  is a tuple of  $m$  variables  $\langle x_1, x_2, \dots, x_m \rangle$ , the constraint scope, and  $\rho$  is a relation of arity  $m$ , the constraint relation. The relation  $\rho$  is a subset of the set of all  $m$ -tuples of elements from the Cartesian product  $D_1 \times D_2 \times \dots \times D_m$  where  $D_i$  is the domain of the variable  $x_i$ :

$$\rho \subseteq \{ \langle d_1, d_2, \dots, d_m \rangle \mid d_1 \in D_1, d_2 \in D_2, \dots, d_m \in D_m \}$$

□

**Constraint Satisfaction Problem.** A CSP is a triple  $P=(X,D,C)$  where  $X$  is a tuple of  $n$  variables  $\langle x_1, x_2, \dots, x_n \rangle$ ,  $D$  is the Cartesian product of the respective domains  $D_1 \times D_2 \times \dots \times D_n$ , i.e. each variable  $x_i$  ranges over the domain  $D_i$ , and  $C$  is a finite set of constraints where the elements of the scope of each constraint are all elements of  $X$ . □

# Basic Concepts

**Constraint Satisfaction.** Let  $P=(X,D,C)$  be a CSP. Let  $(s,\rho)$  be a constraint from  $C$  and  $d$  an element of  $D$ :

$d$  satisfies  $(s,\rho)$  iff  $d[s] \in \rho$



**Solution.** A solution to the CSP  $P=(X,D,C)$  is a tuple  $d \in D$  that satisfies each constraint  $c \in C$ , that is:

$d$  is a solution of  $P$  iff  $\forall c \in C$   $d$  satisfies  $c$



**Consistency.** A CSP  $P=(X,D,C)$  is consistent iff it has at least one solution (otherwise it is inconsistent):

$P$  is consistent iff  $\exists d \in D$   $d$  is a solution of  $P$



# Basic Concepts

**Continuous Constraint Satisfaction Problem.** A CCSP is a CSP  $P=(X,D,C)$  where each domain is an interval of  $\mathbb{R}$  and each constraint relation is defined as a numerical equality or inequality:

- i)  $D=\langle D_1, \dots, D_n \rangle$  where  $D_i$  is a real interval ( $1 \leq i \leq n$ )
- ii)  $\forall c \in C$   $c$  is defined as  $e_c \diamond 0$  where  $e_c$  is a real expression and  $\diamond \in \{\leq, =, \geq\}$   $\square$

**R-interval.** A real interval is a connected set of reals. Let  $a \leq b$  be reals, the following notations for representing real intervals will be used:

- |   |  |
|---|--|
| $[a..b] \equiv \{r \in \mathbb{R} \mid a \leq r \leq b\}$ | $(a..b) \equiv \{r \in \mathbb{R} \mid a < r < b\}$    |
| $(a..b] \equiv \{r \in \mathbb{R} \mid a < r \leq b\}$    | $[a..b) \equiv \{r \in \mathbb{R} \mid a \leq r < b\}$ |
| $[a..+\infty) \equiv \{r \in \mathbb{R} \mid a \leq r\}$  | $(a..+\infty) \equiv \{r \in \mathbb{R} \mid a < r\}$  |
| $(-\infty..b] \equiv \{r \in \mathbb{R} \mid r \leq b\}$  | $(-\infty..b) \equiv \{r \in \mathbb{R} \mid r < b\}$  |
| $(-\infty..+\infty) \equiv \mathbb{R}$                    | $\emptyset \equiv \{\}$                                |

The notation  $\langle a..b \rangle$  will represent a nonempty real interval of any of the defined forms.  $\square$

# Representation of Continuous Domains

## F-Numbers, Intervals and Boxes

**F-numbers.** Let  $F$  be a subset of  $\mathbb{R}$  containing the real number 0 as well as finitely many other reals, and two elements (not reals) denoted by  $-\infty$  and  $+\infty$ :

$$F = \{r_0, \dots, r_n\} \cup \{-\infty, +\infty\} \quad \text{with } 0 \in \{r_0, \dots, r_n\} \subset \mathbb{R}$$

The elements of  $F$  are called  $F$ -numbers. □

$F$  is totally ordered:

any two real elements of  $F$  are ordered as in  $\mathbb{R}$

$-\infty < r < +\infty$  for all real element  $r$

If  $f$  is an  $F$ -number,  $f^-$  and  $f^+$  are the two  $F$ -numbers immediately below and above  $f$  in the total order:

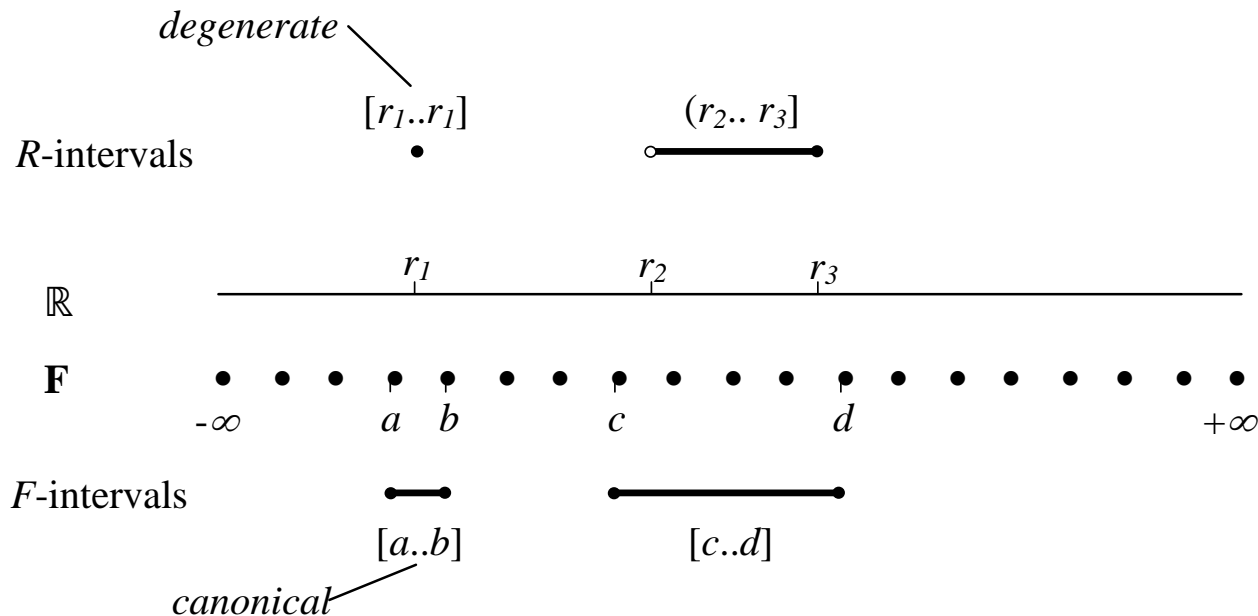
$-\infty^- = -\infty$  and  $+\infty^+ = +\infty$

$-\infty^+$  is the smallest real in  $F$  and  $+\infty^-$  is the largest real in  $F$

# Representation of Continuous Domains

## F-Numbers, Intervals and Boxes

**F-interval.** An *F*-interval is a real interval  $\langle a..b \rangle$  where  $a$  and  $b$  are *F*-numbers.  
In particular, if  $b=a$  or  $b=a^+$  then  $\langle a..b \rangle$  is a *canonical F-interval*. □



In the following we only consider closed *F*-intervals:  $[a,b]$   
If  $a=b$  the interval is degenerated and is represented as  $a$

# Representation of Continuous Domains

## F-Numbers, Intervals and Boxes

Extending the interval concepts to multiple dimensions:

**R-box.** An  $R$ -box  $BR$  with arity  $n$  is the Cartesian product of  $n$   $R$ -intervals and is denoted by  $\langle IR_1, \dots, IR_n \rangle$  where each  $IR_i$  is an  $R$ -interval:

$$BR = \{ \langle r_1, r_2, \dots, r_m \rangle \mid r_1 \in IR_1, r_2 \in IR_2, \dots, r_n \in IR_n \}$$

□

**F-box.** An  $F$ -box  $BF$  with arity  $n$  is the Cartesian product of  $n$   $F$ -intervals and is denoted by  $\langle IF_1, \dots, IF_n \rangle$  where each  $IF_i$  is an  $F$ -interval:

$$BF = \{ \langle r_1, r_2, \dots, r_m \rangle \mid r_1 \in IF_1, r_2 \in IF_2, \dots, r_n \in IF_n \}$$

In particular, if all the  $F$ -intervals  $IF_i$  are canonical then  $BF$  is a **canonical F-box**.

□



# Representation of Continuous Domains

## Interval Operations and Basic Functions

All the usual set operations may also be applied on intervals:

- $\cap$  (intersection)
- $\cup$  (union)
- $\subseteq$  (inclusion)

A particularly useful operation is the union hull ( $\uplus$ ):

**Union Hull.** Let  $I_1 = \langle a_1..b_1 \rangle_1$  and  $I_2 = \langle a_2..b_2 \rangle_2$  be two intervals. The union hull operation ( $\uplus$ ) is defined as:

$$I_1 \uplus I_2 = \begin{cases} I_1 \cup I_2 & \text{if } I_1 \cap I_2 \neq \emptyset \\ \langle a_1..b_2 \rangle_2 & \text{if } \forall r_1 \in I_1 \forall r_2 \in I_2 \ r_1 < r_2 \\ \langle a_2..b_1 \rangle_1 & \text{if } \forall r_1 \in I_1 \forall r_2 \in I_2 \ r_2 < r_1 \end{cases}$$



In the case of closed intervals  $[a,b]$  and  $[c,d]$ :

$$[a,b] \uplus [c,d] = [\min(a,c), \max(b,d)]$$

# Representation of Continuous Domains

## Interval Operations and Basic Functions

**Interval Basic Functions.** Let  $[a..b]$  be a closed interval. The following basic functions return a real value and are defined as:

$$\text{left}([a..b]) = a$$

$$\text{right}([a..b]) = b$$

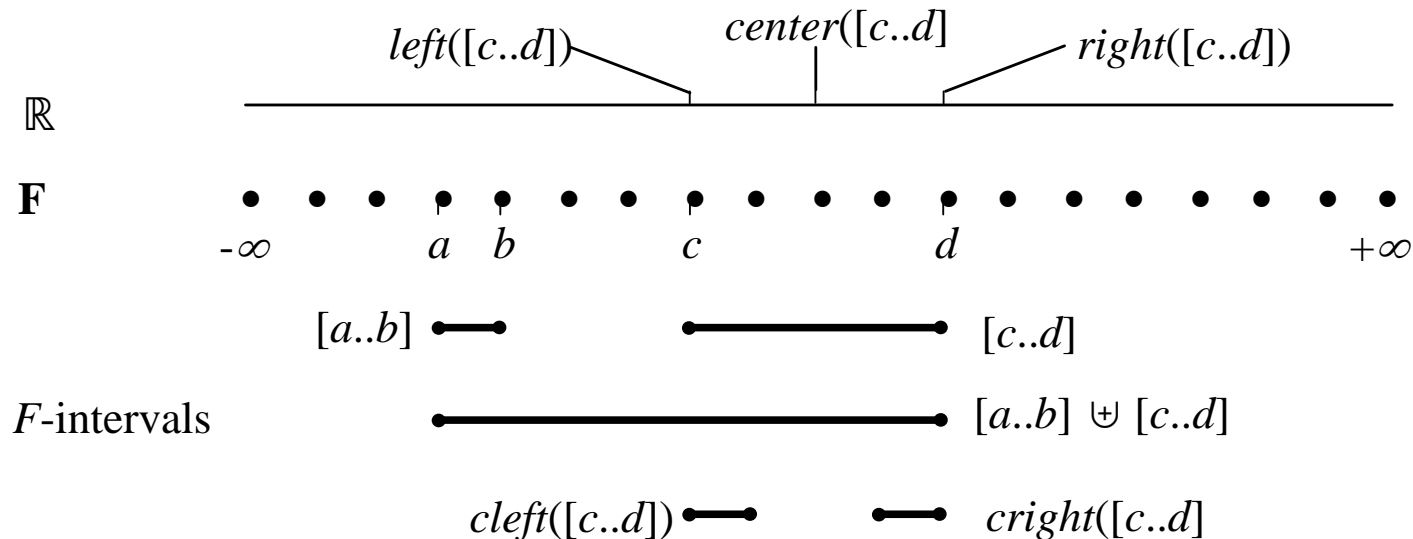
$$\text{center}([a..b]) = (a+b)/2$$

$$\text{width}([a..b]) = b-a$$

Let  $[a..b]$  be a closed  $F$ -interval. The following basic functions return a canonical  $F$ -interval and are defined as:

$$\text{cleft}([a..b]) = \begin{cases} [a] & \text{if } a=b \\ [a..a^+] & \text{if } a < b \end{cases}$$

$$\text{cright}([a..b]) = \begin{cases} [b] & \text{if } a=b \\ [b^-..b] & \text{if } a < b \end{cases} \quad \square$$



# Representation of Continuous Domains

## Interval Approximations

For any real number  $r$  we will denote by:

$\lfloor r \rfloor$  the largest  $F$ -number not greater than  $r$        $(\lfloor -\infty \rfloor = -\infty)$

$\lceil r \rceil$  the smallest  $F$ -number not smaller than  $r$        $(\lceil +\infty \rceil = +\infty)$

**Interval Approximation.** Let  $IR = \langle a..b \rangle$  be a real interval. The interval approximation of  $IR$ , denoted  $I_{apx}(IR)$ , is the smallest  $F$ -interval including  $IR$  ( $IR \subseteq I_{apx}(IR)$ ):

$$I_{apx}(IR) = [\lfloor a \rfloor .. \lceil b \rceil].$$

In the special case where  $IR$  is a single real  $\{r\} = [r..r]$  then  $I_{apx}(IR) = [\lfloor r \rfloor .. \lceil r \rceil]$ . □

**Set Approximation.** Let  $SR$  be a set of real values defined by the union of  $n$  real intervals ( $SR = IR_1 \cup \dots \cup IR_n$ ). The set approximation of  $SR$ , denoted  $S_{apx}(SR)$ , is the set defined by the union of the  $n$  corresponding interval approximations:

$$S_{apx}(SR) = I_{apx}(IR_1) \cup \dots \cup I_{apx}(IR_n) \quad \square$$

**Hull Approximation.** Let  $SR$  be a set of real values defined by the union of  $n$  real intervals ( $SR = IR_1 \cup \dots \cup IR_n$ ). The hull approximation of  $SR$ , denoted  $I_{hull}(SR)$ , is the  $F$ -interval defined by:

$$I_{hull}(SR) = I_{apx}(IR_1) \uplus \dots \uplus I_{apx}(IR_n) \quad \square$$

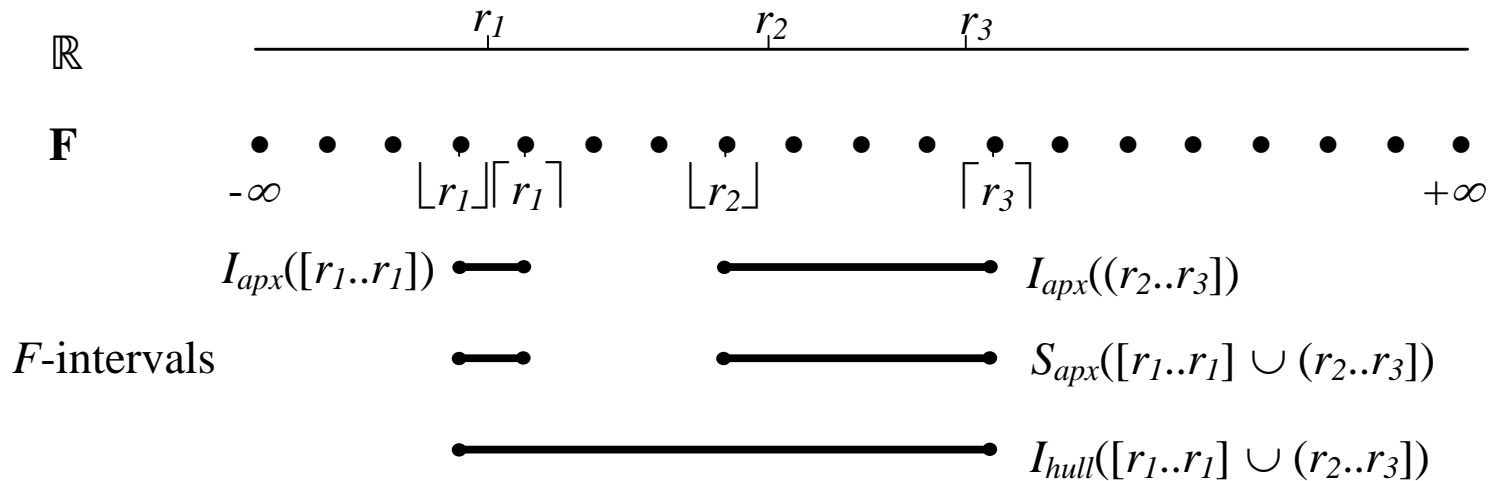
# Representation of Continuous Domains

## Interval Approximations

For any real number  $r$  we will denote by:

$$\lfloor r \rfloor \text{ the largest } F\text{-number not greater than } r \quad (\lfloor -\infty \rfloor = -\infty)$$

$$\lceil r \rceil \text{ the smallest } F\text{-number not smaller than } r \quad (\lceil +\infty \rceil = +\infty)$$



# Interval Arithmetic

Interval arithmetic is an extension of real arithmetic for intervals

## Basic Interval Arithmetic Operators

The basic operators are redefined for intervals:

the result is the set obtained by applying the operator to any pair of reals from the interval operands

**Basic Interval Arithmetic Operators.** Let  $I_1$  and  $I_2$  be two real intervals (bounded and closed). The basic arithmetic operations on intervals are defined by:

$$I_1 \Phi I_2 = \{ r_1 \Phi r_2 \mid r_1 \in I_1 \wedge r_2 \in I_2 \} \quad \text{with} \quad \Phi \in \{+, -, \times, /\}$$

except that  $I_1/I_2$  is not defined if  $0 \in I_2$ . □

Algebraic rules may be defined to evaluate any basic operation on intervals in terms of formulas for its bounds

**Evaluation Rules for the Basic Operators.** Let  $[a..b]$  and  $[c..d]$  be two real intervals (bounded and closed):

$$[a..b] + [c..d] = [a+c..b+d] \qquad [a..b] - [c..d] = [a-d..b-c]$$

$$[a..b] \times [c..d] = [\min(ac, ad, bc, bd).. \max(ac, ad, bc, bd)]$$

$$[a..b] / [c..d] = [a..b] \times [1/d..1/c] \qquad \text{if } 0 \notin [c..d] \quad \square$$

# Interval Arithmetic

## Algebraic Properties

Most algebraic properties of real arithmetic also hold for interval arithmetic: the distributive law is an exception

**Algebraic Properties of the Basic Operators.** Let  $I_1, I_2, I_3$  and  $I_4$  be real intervals (bounded and closed). The following algebraic properties hold for the basic interval operations:

Commutativity:	$I_1 + I_2 = I_2 + I_1$	(interval addition)
	$I_1 \times I_2 = I_2 \times I_1$	(interval multiplication)
Associativity:	$(I_1 + I_2) + I_3 = I_1 + (I_2 + I_3)$	(interval addition)
	$(I_1 \times I_2) \times I_3 = I_1 \times (I_2 \times I_3)$	(interval multiplication)
Neutral Element:	$I_1 + [0..0] = I_1$	(interval addition)
	$I_1 \times [1..1] = I_1$	(interval multiplication)
Subdistributivity:	$I_1 \times (I_2 + I_3) \subseteq I_1 \times I_2 + I_1 \times I_3$	
Inclusion Monotonicity:	$I_1 \subseteq I_3 \wedge I_2 \subseteq I_4 \Rightarrow I_1 \Phi I_2 \subseteq I_3 \Phi I_4$	
	(with: $\Phi \in \{+, -, \times, /\}$ and $I_3 \Phi I_4$ defined)	□

Inclusion monotonicity is an important new concept

# Interval Arithmetic

## Algebraic Properties

### Example of Subdistributivity:

$$\begin{array}{l}
 I_1=[0..1] \\
 I_2=[2..3] \\
 I_3=[-2..-1]
 \end{array}
 \left|
 \begin{array}{l}
 I_1 \times (I_2 + I_3) \quad \subseteq \\
 [0..1] \times ([2..3] + [-2..-1]) \\
 \underbrace{[0..1] \times [0..2]}_{[0..2]} \\
 \subseteq
 \end{array}
 \begin{array}{l}
 I_1 \times I_2 + I_1 \times I_3 \\
 [0..1] \times [2..3] + [0..1] \times [-2..-1] \\
 \underbrace{[0..3]} + \underbrace{[-2..0]} \\
 [-2..3]
 \end{array}$$

### Example of Inclusion monotonicity: (the same operations with smaller domains)

$$\begin{array}{l}
 I_1=[0.5..1] \\
 I_2=[2..2.5] \\
 I_3=[-2..-1]
 \end{array}
 \left|
 \begin{array}{l}
 I_1 \times (I_2 + I_3) \quad \subseteq \\
 [0.5..1] \times ([2..2.5] + [-2..-1]) \\
 \underbrace{[0.5..1] \times [0..1.5]}_{[0..1.5]} \\
 \subseteq
 \end{array}
 \begin{array}{l}
 I_1 \times I_2 + I_1 \times I_3 \\
 [0.5..1] \times [2..2.5] + [0.5..1] \times [-2..-1] \\
 \underbrace{[1..2.5]} + \underbrace{[-2..-0.5]} \\
 [-1..2]
 \end{array}$$

# Interval Arithmetic

## Safe Evaluation

In interval arithmetic computations of the correct real values must be always within the bounds of the resulting interval

Outward rounding forces the result of any basic interval arithmetic operation to be the interval approximation of the correct real interval (obtained with infinite precision)

**Outward Rounding Evaluation Rules of the Basic Operators.** Let  $[a..b]$  and  $[c..d]$  be two  $F$ -intervals (bounded and closed):

$$\begin{aligned} [a..b] + [c..d] &= [\lfloor a+c \rfloor, \lceil b+d \rceil] & [a..b] - [c..d] &= [\lfloor a-d \rfloor, \lceil b-c \rceil] \\ [a..b] \times [c..d] &= [\min(\lfloor ac \rfloor, \lfloor ad \rfloor, \lfloor bc \rfloor, \lfloor bd \rfloor), \max(\lceil ac \rceil, \lceil ad \rceil, \lceil bc \rceil, \lceil bd \rceil)] \\ [a..b] / [c..d] &= [a..b] \times [\lfloor 1/d \rfloor, \lceil 1/c \rceil] & \text{if } 0 \notin [c..d] \end{aligned}$$

If  $\Phi$  is a basic interval arithmetic operator then  $\Phi_{apx}$  denotes the corresponding outward evaluation rule:  $\Phi_{apx}(I_1, \dots, I_m) = I_{apx}(\Phi(I_1, \dots, I_m))$



# Interval Arithmetic

## Safe Evaluation

In interval arithmetic computations the correct real values must be always within the bounds of the resulting interval

The correctness of the interval arithmetic computations is guaranteed by the inclusion monotonicity property:

if the correct real values are within the operand intervals then the correct real values resulting from any interval arithmetic operation must also be within the resulting interval.

The computation of a successive composition of basic arithmetic operations over real intervals preserve the correct real values within the final resulting interval

# Interval Arithmetic

## Extended Interval Arithmetic

Extensions on the definition of the division operator:

allow division by an interval containing 0

if  $c < 0 < d$  then  $[a,b]/[c,d] = [a,b]/[c,0^-] \cup [a,b]/[0^+,d]$

$$[1,2]/[-1,1] = [1,2]/[-1,0^-] \cup [1,2]/[0^+,1]$$

$$[-\infty,-1] \cup [1,+\infty]$$

Extensions on the real intervals allowed as arguments:

allow open intervals and infinite bounds

$$(-\infty,-1] + [-1,3] = (-\infty,2]$$

$$(-\infty,-1] + [-1,+\infty] = (-\infty,+\infty]$$

Extensions on the set of basic interval operators:

allow other elementary functions (*exp, ln, power, sin, cos...*)

$$\exp([a,b]) = [\exp(a), \exp(b)]$$

# Interval Functions

## Interval Expressions and their Evaluation

**Real and Interval Expressions.** An expression  $E$  is an inductive structure defined in the following way:

- (i) a constant is an expression;
- (ii) a variable is an expression;
- (iii) if  $E_1, \dots, E_m$  are expressions and  $\Phi$  is a  $m$ -ary basic operator then  $\Phi(E_1, \dots, E_m)$  is an expression;

A real expression is an expression with real constants, real valued variables and real operators. An interval expression is an expression with interval constants, interval valued variables and interval operators. □

If  $x_1$ ,  $x_2$  and  $x_3$  are real valued variables then  $(x_1+x_2) \times (x_3-\pi)$  is a real expression with three binary real operators (+,  $\times$  and -) and a real constant ( $\pi$ ).

If  $X_1$  and  $X_2$  are interval valued variables then  $(X_1 + \cos([0.. \pi] \times X_2))$  is an interval expression with two binary interval operators (+ and  $\times$ ), a unary interval operator ( $\cos$ ) and an interval constant ( $[0.. \pi]$ ).

# Interval Functions

## Interval Expressions and their Evaluation

Interval arithmetic provides a safe method for evaluating an interval expression:

- replace each variable by its interval domain;
- apply recursively the basic operator evaluation rules

**Evaluation of an Interval Expression.** Let  $F$  be the  $n$ -ary interval function represented by the interval expression  $F_E$ , and  $B$  an  $n$ -ary  $R$ -box. The interval arithmetic evaluation of  $F_E$  wrt  $B$  is an interval function recursively defined as:

$$F_E(B) = \begin{cases} I_{\text{apx}}(I) & \text{if } F_E \equiv I & (I \text{ is an interval constant}) \\ I_{\text{apx}}(B[X_i]) & \text{if } F_E \equiv X_i & (X_i \text{ is an interval variable}) \\ \Phi_{\text{apx}}(E_1(B), \dots, E_m(B)) & \text{if } F_E \equiv \Phi(E_1, \dots, E_m) & (\Phi \text{ is an interval operator}) \quad \square \end{cases}$$

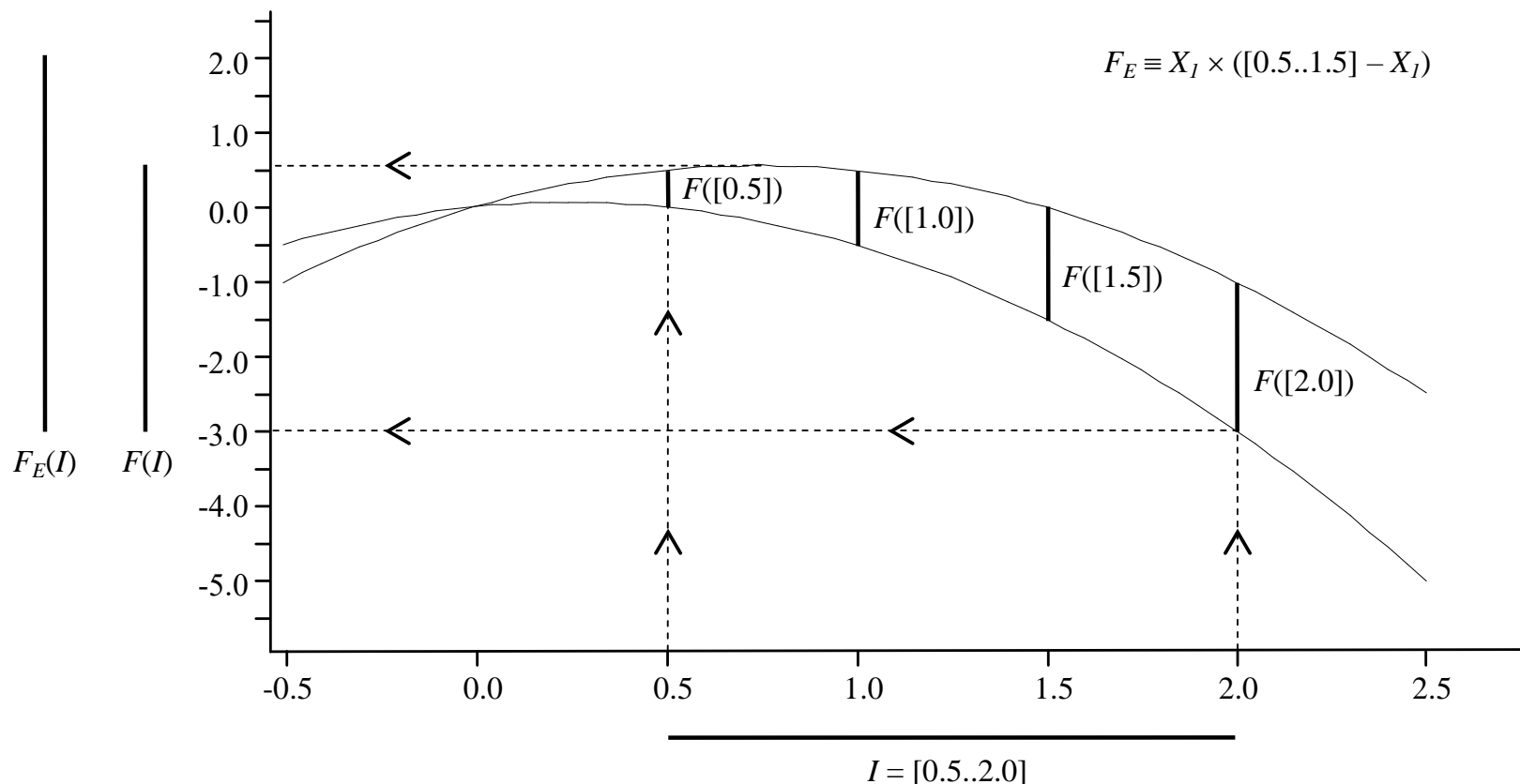
The interval arithmetic evaluation of an interval expression provides a sound computation of the interval function represented by the expression

# Interval Functions

## Interval Expressions and their Evaluation

**Soundness of the Interval Expression Evaluation.** Let  $F_E$  be an interval expression representing the  $n$ -ary interval function  $F$ , and  $B$  an  $n$ -ary  $R$ -box. The interval arithmetic evaluation of  $F_E$  with respect to  $B$  is sound:

$$F(B) \subseteq F_E(B)$$



# Interval Functions

## Interval Extensions

**Interval Extension of a Real Function.** Let  $f$  be an  $n$ -ary real function with domain  $D_f$ , and  $F$  an  $n$ -ary interval function. The interval function  $F$  is an interval extension of the real function  $f$  iff:

$$\forall \langle r_1, \dots, r_n \rangle \in D_f \quad f(\langle r_1, \dots, r_n \rangle) \in F(\langle [r_1..r_1], \dots, [r_n..r_n] \rangle) \quad \square$$

If  $F$  is an interval extension of  $f$  then each real value mapped by  $f$  must lie within the interval mapped by  $F$  when the argument is the corresponding box of degenerate intervals

Consequently,  $F$  provides a sound evaluation of  $f$  in the sense that the correct real value is not lost

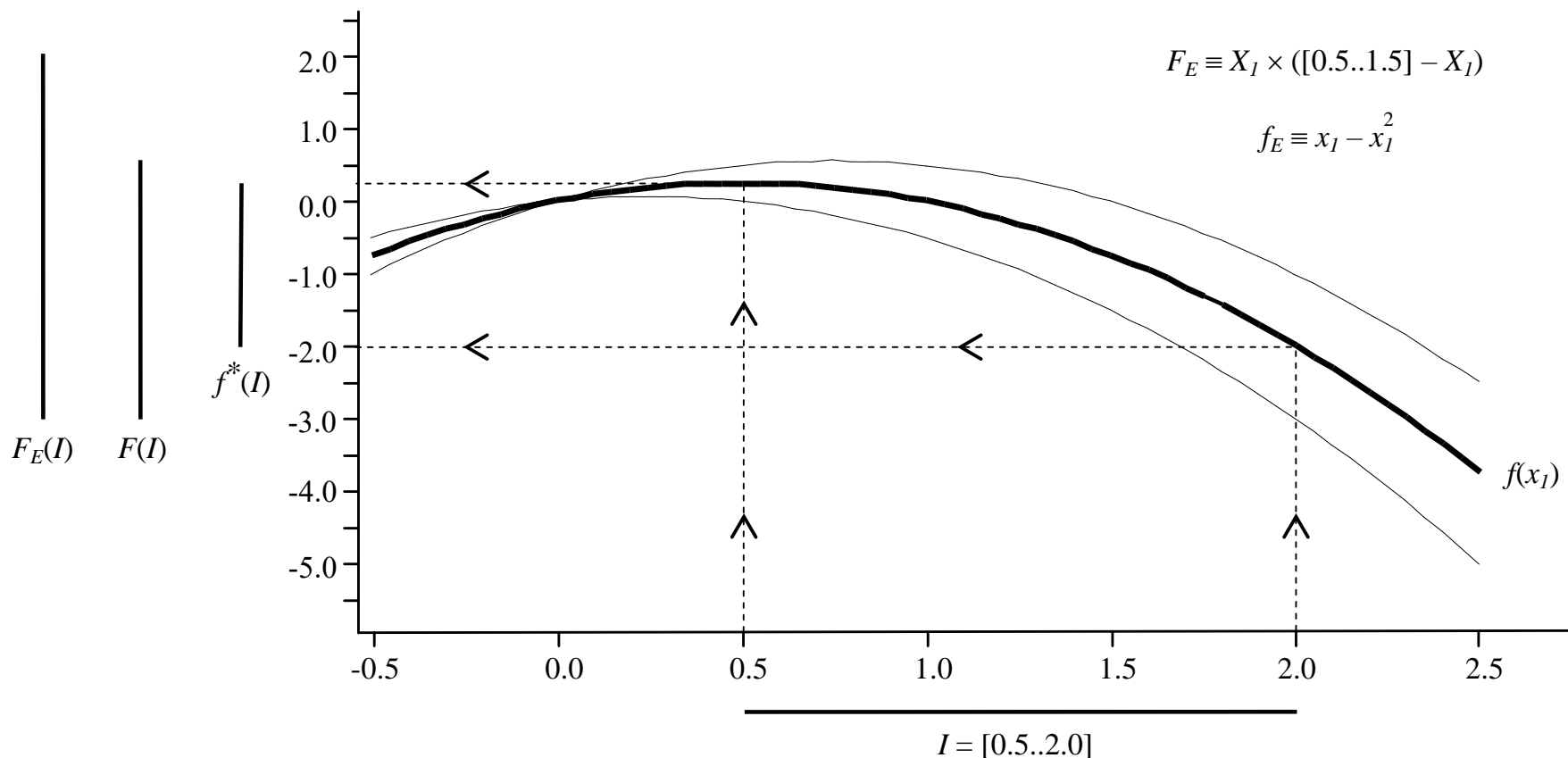
The interval arithmetic evaluation of any expression representing an interval extension of a real function provides a sound evaluation for its range and is itself an interval extension of the real function

# Interval Functions

## Interval Extensions

**Soundness of the Evaluation of an Interval Extension.** Let  $F$  be an interval extension of an  $n$ -ary real function  $f$ ,  $F_E$  an interval expression representing  $F$ , and  $B$  be  $n$ -ary  $R$ -box. Then, both  $F(B)$  and  $F_E(B)$ , enclose the range of  $f$  over  $B$ :

$$f^*(B) \subseteq F(B) \subseteq F_E(B)$$



# Interval Functions

## Interval Extensions

**Natural Interval Expression.** If  $f_E$  is a real expression representing the real function  $f$ , then its natural interval expression  $F_n$  is obtained by replacing in  $f_E$ : each real variable  $x_i$  by an interval variable  $X_i$ ; each real constant  $k$  by the real interval  $[k..k]$ , and each real operator by the corresponding interval operator.  $\square$

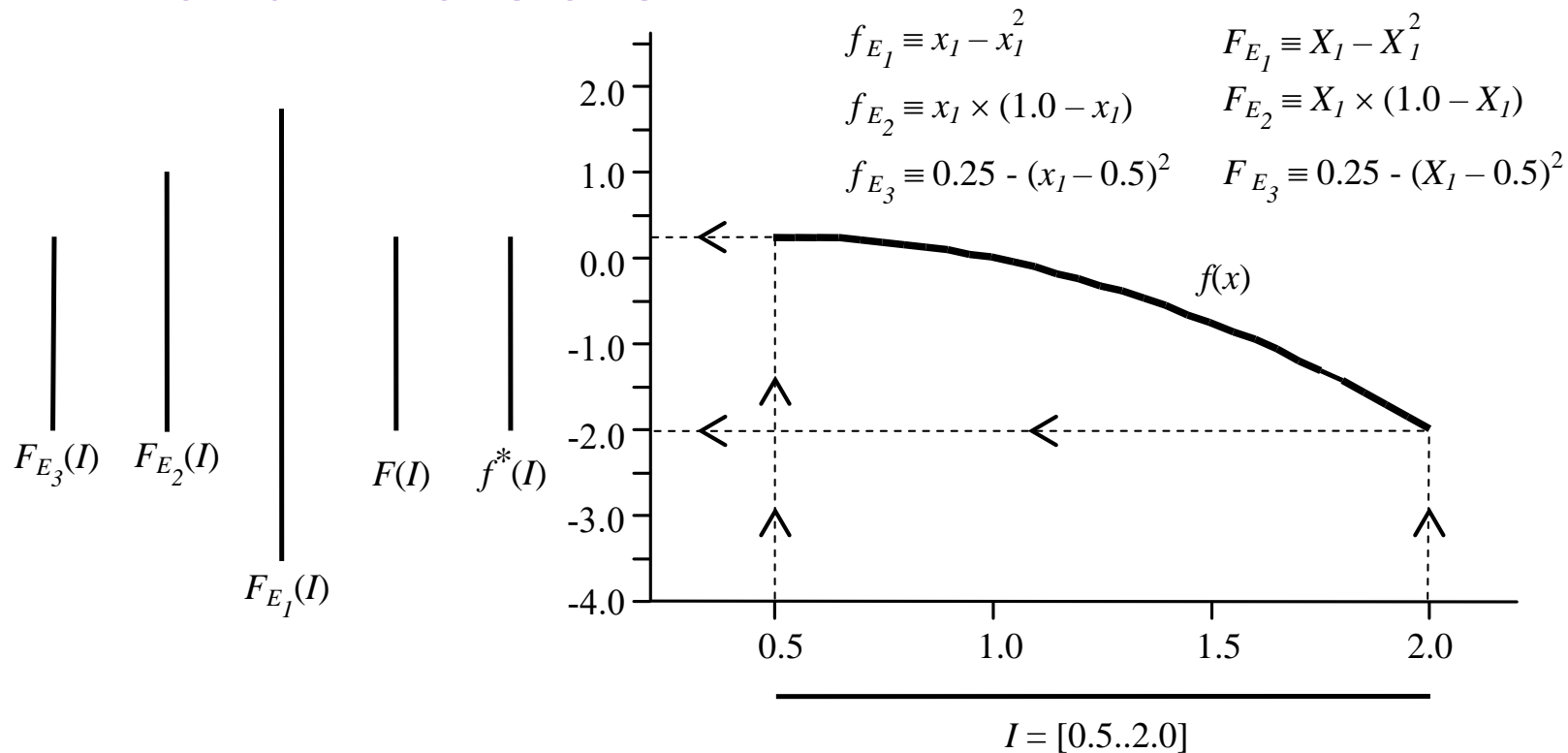
**Natural Interval Extension.** Let  $f_E$  be a real expression representing the real function  $f$ , and  $F_n$  be the natural interval expression of  $f_E$ . The interval function  $F$  represented by  $F_n$  is the smallest interval enclosure for the range of  $f$  and the interval arithmetic evaluation of  $F_n$  is an interval extension of  $f$  denominated Natural interval extension w.r.t.  $f_E$ .  $\square$

Several equivalent real expressions may represent the same real function  $f$ . Consequently, the natural interval extensions with respect to these equivalent real expressions are all interval extensions of  $f$ .



# Interval Functions

## Interval Extensions



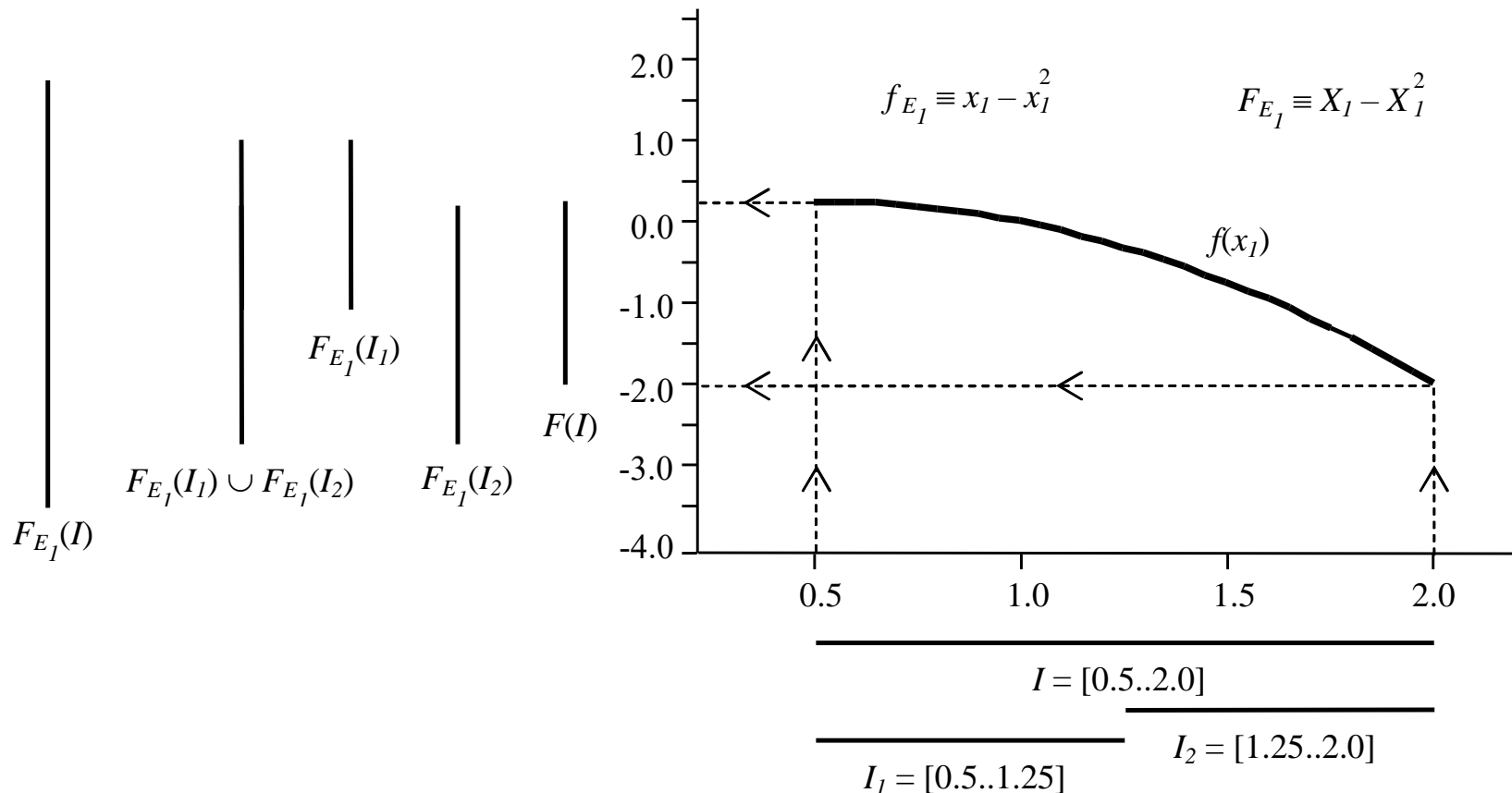
**Intersection of Interval Extensions.** Let  $F_1$  and  $F_2$  be two  $n$ -ary interval functions and  $B$  an  $n$ -ary  $R$ -box. Let  $F$  be an  $n$ -ary interval function defined by:  $F(B) = F_1(B) \cap F_2(B)$ . If  $F_1$  and  $F_2$  are interval extensions of the real function  $f$ , then  $F$  is also an interval extension of  $f$ . □

# Interval Functions

## Interval Extensions

**Decomposed Evaluation of an Interval Extension.** Let  $F$  be an interval extension of the  $n$ -ary real function  $f$ , and  $F_E$  an interval expression representing  $F$ . Let  $B$ ,  $B_1$  and  $B_2$  be  $n$ -ary  $R$ -boxes. If  $B=B_1 \cup B_2$  then:

$$F(B) \subseteq F_E(B_1) \cup F_E(B_2) \subseteq F_E(B)$$



# Interval Functions

## Interval Extensions

**Dependency Problem.** In the interval arithmetic evaluation of an interval expression, each occurrence of the same variable is treated as a different variable. The dependency between the different occurrences of a variable in an expression is lost.  $\square$

**No Overestimation Without Multiple Variable Occurrences.** Let  $F_E$  be an interval expression representing the  $n$ -ary interval function  $F$ , and  $B$  an  $n$ -ary  $R$ -box. If  $F_E$  is an interval expression in which each variable occurs only once then:  
 $F(B) = F_E(B)$  (w/ exact interval operators and infinite precision arithmetic evaluation)  $\square$

