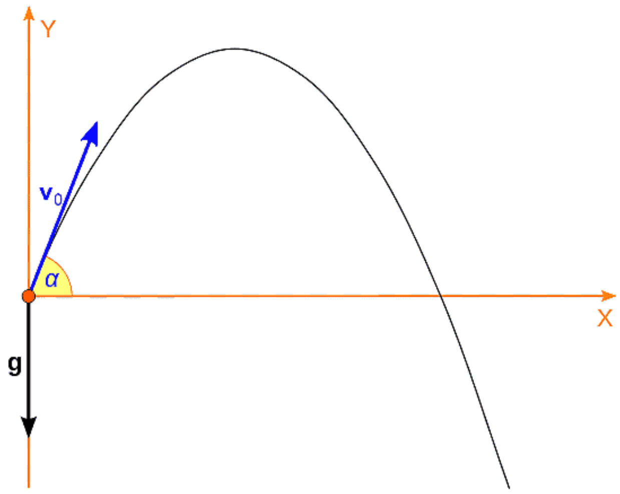
Constraint Programming

2021/2022 – Mini-Test #2

Monday, 3 January, 9:00 h in 127-Ed.II

Duration: 1.5 h (open book)

The motion of a projectile through the air that is subject only to the acceleration of gravity (g=9.81 ms^{-2}) is represented in the figure bellow.



Given the launch angle α (radians) and the initial velocity v_0 (ms^{-1}), the projectile follows a parabolic trajectory represented by the following equation:

$$y = (\tan\alpha)x - \frac{g}{2v_0^2(\cos\alpha)^2}x^2$$

1. Interval Arithmetic

The horizontal range *r* of the projectile is the horizontal distance it has traveled when it returns to its initial height (*y*=0). Given the initial velocity v_0 and solving the above equation with respect to *x*, the horizontal range *r* can be computed as a function of α:

$$r = f(\alpha) = \frac{2v_0^2}{g}(\sin\alpha)(\cos\alpha)$$

In the following questions assume that $v_0 = 10$ ms^{-1} and g=9.81 ms^{-2}.

1.1. Define $F_n(I)$ the natural interval extension of *f*.

$$F_n(I) = \frac{2(10)^2}{9.81}(\sin I)(\cos I) = [20.3874](\sin I)(\cos I)$$

1.2. Define $F_c(I)$ the mean value extension of *f* over the interval $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ centered at the midpoint.

Some derivative rules:

$$\big(f(x) \times g(x)\big)' = f(x) \times \big(g(x)\big)' + \big(f(x)\big)' \times g(x)$$
$$(\sin x)' = (\cos x)$$
$$(\cos x)' = -(\sin x)$$

$$F_c(x) = f(c) + F'([a,b])(x-c) \quad \text{with} \quad c = \frac{a+b}{2}$$

$$[a,b] = \left[\frac{\pi}{6}, \frac{\pi}{3}\right] \qquad c = \frac{\pi}{4}$$

$$f(c) = \frac{2(10)^2}{9.81}\left(\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{4}\right) = \frac{2(10)^2}{9.81}\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{10^2}{9.81} = [10.1937]$$

$$F'(I) = \left(\frac{2(10)^2}{9.81}(\sin I)(\cos I)\right)' = \frac{2(10)^2}{9.81}\big((\sin I)(\cos I)\big)'$$

$$F'(I) = \frac{2(10)^2}{9.81}((\sin I)'(\cos I) + (\sin I)(\cos I)') = \frac{2(10)^2}{9.81}((\cos I)^2 - (\sin I)^2)$$

$$F'\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right) = \frac{2(10)^2}{9.81}\left(\left(\cos\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right)^2 - \left(\sin\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right)^2\right) = \frac{2(10)^2}{9.81}\left(\left(\left[\frac{1}{2},\frac{\sqrt{3}}{2}\right]\right)^2 - \left(\left[\frac{1}{2},\frac{\sqrt{3}}{2}\right]\right)^2\right)$$

$$F'\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right) = \frac{2(10)^2}{9.81}\left(\left[\frac{1}{4},\frac{3}{4}\right] - \left[\frac{1}{4},\frac{3}{4}\right]\right) = \frac{2(10)^2}{9.81}\left(\left[\frac{1}{4}-\frac{3}{4},\frac{3}{4}-\frac{1}{4}\right]\right) = \frac{2(10)^2}{9.81}\left(\left[-\frac{1}{2},\frac{1}{2}\right]\right)$$

$$F'\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right) = \frac{10^2}{9.81}([-1,1])=[-10.1937,10.1937]$$

$$F_c(I) = \frac{10^2}{9.81} + \frac{10^2}{9.81}([-1,1])\left(I - \frac{\pi}{4}\right) = [10.1937] + [-10.1937,10.1937]\left(I - \frac{\pi}{4}\right)$$

1.3. Given that: $2(\sin\alpha)(\cos\alpha) = \sin 2\alpha$

Define $F_r(I)$ that computes the sharpest enclosure of the range of f for any $I \subseteq \left[\frac{\pi}{6},\frac{\pi}{3}\right]$.

$$\frac{2v_0^2}{g}(\sin\alpha)(\cos\alpha) = \frac{v_0^2}{g}(\sin 2\alpha)$$

$$F_r(I) = \frac{10^2}{9.81}(\sin 2I)$$

With $I = [a,b] \subseteq \left[\frac{\pi}{6},\frac{\pi}{3}\right]$:

$$F_r([a,b]) = \begin{cases} \frac{10^2}{9.81}[\sin 2a, \sin 2b] & if\ 2b \le \frac{\pi}{2} \\ \frac{10^2}{9.81}[min(\sin 2a, \sin 2b), 1] & if\ 2a < \frac{\pi}{2} < 2b \\ \frac{10^2}{9.81}[\sin 2a, \sin 2b] & if\ 2a \ge \frac{\pi}{2} \end{cases}$$

1.4. Compute $F_n\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right)$, $F_c\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right)$ and $F_r\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right)$.

$$F_n\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right) = \frac{2(10)^2}{9.81}\left(\sin\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right)\left(\cos\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right) = \frac{2(10)^2}{9.81}\left[\frac{1}{2},\frac{\sqrt{3}}{2}\right]\left[\frac{1}{2},\frac{\sqrt{3}}{2}\right] = \frac{2(10)^2}{9.81}\left[\frac{1}{4},\frac{3}{4}\right]$$

$$F_n\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right) = [5.09684,15.2905]$$

$$F_c\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right) = [10.1937] + [-10.1937,10.1937]\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right] - \frac{\pi}{4}\right)$$

$$F_c\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right) = [10.1937] + [-10.1937,10.1937]\left(\left[-\frac{\pi}{12},\frac{\pi}{12}\right]\right) = [10.1937] + [-2.6687,2.6687]$$

$$F_c\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right) = [7.52498,12.8624]$$

$$F_r\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right) = \frac{10^2}{9.81}\left[min\left(\sin 2\frac{\pi}{6}, \sin 2\frac{\pi}{3}\right), 1\right] \qquad \left(\text{since: } 2\frac{\pi}{6} < \frac{\pi}{2} < 2\frac{\pi}{3}\right)$$

$$F_r\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right) = \frac{10^2}{9.81}\left[min\left(\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right),1\right] = \frac{10^2}{9.81}\left[\frac{\sqrt{3}}{2},1\right] = [8.82799,10.1937]$$

1.5. From the results obtained in 1.4 what can you conclude about the truth value of the following propositions? (Justify your answers)

a. $\exists_{\frac{\pi}{6}\le\alpha\le\frac{\pi}{3}} f(\alpha) = 8$

False: F_r is an extension of f thus $\forall_{\frac{\pi}{6}\le\alpha\le\frac{\pi}{3}} f(\alpha) \in F_r\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right)$.

Since $8 \notin F_r\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right) = [8.82799,10.1937]$ then $\forall_{\frac{\pi}{6}\le\alpha\le\frac{\pi}{3}} f(\alpha) \neq 8$

b. $\exists_{\frac{\pi}{6}\le\alpha\le\frac{\pi}{3}} f(\alpha) = 10$

True: F_r is a sharp extension of f thus: $\exists_{\frac{\pi}{6}\le\alpha_1,\alpha_2\le\frac{\pi}{3}} f(\alpha_1) = 8.82799 \wedge f(\alpha_2) = 10.1937$

Since f is continuous then $\exists_{\frac{\pi}{6}\le\min(\alpha_1,\alpha_2)\le\alpha\le\max(\alpha_1,\alpha_2)\le\frac{\pi}{3}} f(\alpha) = 10$

2. Interval Newton

Consider the function: $h(\alpha) = r - \frac{2v_0^2}{g}(\sin\alpha)(\cos\alpha)$

Notice that when $h(\alpha) = 0$, the projectile launched with an angle α radians and initial velocity v_0 ms^{-1}, reach the horizontal range r.

2.1. Define the interval Newton function with respect to h (do not assign values to v_0 and r).

$$N([a,b]) = c - \frac{f(c)}{F'([a,b])} \qquad with \quad c = \frac{a+b}{2}$$

$$f(c) = r - \frac{2v_0^2}{9.81}(\sin c)(\cos c)$$

$$F'([a,b]) = -\frac{2v_0^2}{9.81}\left((\cos[a,b])^2 - (\sin[a,b])^2\right)$$

$$N([a,b]) = c - \frac{r - \frac{2v_0^2}{9.81}(\sin c)(\cos c)}{-\frac{2v_0^2}{9.81}((\cos[a,b])^2 - (\sin[a,b])^2)} \qquad with \quad c = \frac{a+b}{2}$$

$$N([a,b]) = c + \frac{\frac{9.81}{2v_0^2}r - (\sin c)(\cos c)}{(\cos[a,b])^2 - (\sin[a,b])^2} \qquad with \quad c = \frac{a+b}{2}$$

2.2. Use the Newton function to prove that with an initial velocity of 10 ms^{-1} there is no angle α between $\frac{\pi}{6}$ and $\frac{\pi}{3}$ to obtain a trajectory range smaller than 5 m.

$$N\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right) = \frac{\pi}{4} + \frac{\frac{9.81}{200}[0,5] - \left(\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{4}\right)}{\left(\cos\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right)^2 - \left(\sin\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right)^2} = \frac{\pi}{4} + \frac{[0,0.24525] - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}{\left(\left[\frac{1}{2},\frac{\sqrt{3}}{2}\right]\right)^2 - \left(\left[\frac{1}{2},\frac{\sqrt{3}}{2}\right]\right)^2}$$

$$N\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right) = \frac{\pi}{4} + \frac{[0,0.24525] - \left(\frac{1}{2}\right)}{\left[\frac{1}{4},\frac{3}{4}\right] - \left[\frac{1}{4},\frac{3}{4}\right]} = \frac{\pi}{4} + \frac{[-0.5,-0.25475]}{\left[-\frac{1}{2},\frac{1}{2}\right]} = \frac{\pi}{4} - \frac{[0.5095,1]}{[-1,1]}$$

$$N\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right) = \frac{\pi}{4} - ([-\infty,-0.5095] \cup [0.5095,+\infty]) = \left[-\infty,\frac{\pi}{4} - 0.5095\right] \cup \left[\frac{\pi}{4} + 0.5095,+\infty\right]$$

$$N\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right) = [-\infty,0.2759] \cup [1.2949,+\infty]$$

$$\left[\frac{\pi}{6},\frac{\pi}{3}\right] = [0.5236,1.0472]$$

$N\left(\left[\frac{\pi}{6},\frac{\pi}{3}\right]\right) \cap \left[\frac{\pi}{6},\frac{\pi}{3}\right] = \emptyset$ implies that there are no roots in $\left[\frac{\pi}{6},\frac{\pi}{3}\right]$ so there is no angle α between π/6 and π/3 to obtain a trajectory range smaller than 5 m.

2.3. Use the Newton function to prove that with an initial velocity of 10 ms^{-1} there is an angle α between $\frac{\pi}{6}$ and $\frac{\pi}{5}$ to obtain a trajectory range of 9 m and compute an interval enclosure of such angle with width not larger than 0.02 radians.

$$N\left(\left[\frac{\pi}{6},\frac{\pi}{5}\right]\right) = \frac{11\pi}{60} + \frac{\frac{9.81}{200}9 - \left(\sin\frac{11\pi}{60}\right)\left(\cos\frac{11\pi}{60}\right)}{\left(\cos\left[\frac{\pi}{6},\frac{\pi}{5}\right]\right)^2 - \left(\sin\left[\frac{\pi}{6},\frac{\pi}{5}\right]\right)^2} = 0.575959 + \frac{0.44145 - (0.544639)(0.83867)}{\left(\left[0.80902,\frac{\sqrt{3}}{2}\right]\right)^2 - \left(\left[\frac{1}{2},0.5878\right]\right)^2}$$

$$N\left(\left[\frac{\pi}{6},\frac{\pi}{5}\right]\right) = 0.575959 + \frac{0.44145 - 0.456773}{[0.654508,0.75] - [0.25,0.345492]} = 0.575959 + \frac{-0.015323}{[0.309016,0.5]}$$

$$N\left(\left[\frac{\pi}{6},\frac{\pi}{5}\right]\right) = 0.575959 - [0.030646,0.049586] = [0.526373,0.545313]$$

Since $N\left(\left[\frac{\pi}{6},\frac{\pi}{5}\right]\right) = [0.526373,0.545313] \subset \left[\frac{\pi}{6},\frac{\pi}{5}\right] = [0.5236,0.62832] \Rightarrow$ There is at least one root in [0.526373,0.545313] (width 0.01894 < 0.02)

3. Constraint Propagation

Consider the constraint below obtained from the trajectory equation with angle α = $\frac{\pi}{4}$ radians:

$$y = x - 9.81\left(\frac{x}{v_0}\right)^2$$

3.1. Is the constraint box-consistent in box $B[x, y, v_0]$ = [4,5]×[2,3]×[9,10]?

box-consistent in [4,5]×[2,3]×[9,10]

$$\Leftrightarrow$$

$$0 \in [2,3] + 9.81\left(\frac{4}{[9,10]}\right)^2 - 4 = [-0.4304, 0.937778]$$

$$\wedge \quad 0 \in [2,3] + 9.81\left(\frac{5}{[9,10]}\right)^2 - 5 = [-0.5475, 1.02778]$$

$$\wedge \quad 0 \in 2 + 9.81\left(\frac{[4,5]}{[9,10]}\right)^2 - [4,5] = [-1.4304, 1.02778]$$

$$\wedge \quad 0 \in 3 + 9.81\left(\frac{[4,5]}{[9,10]}\right)^2 - [4,5] = [-0.4304, 2.02778]$$

$$\wedge \ 0 \in [2,3] + 9.81\left(\frac{[4,5]}{9}\right)^2 - [4,5] = [-1.06222, 2.02778]$$

$$\wedge \quad 0 \in [2,3] - 9.81\left(\frac{[4,5]}{10}\right)^2 - [4,5] = [-1.4304, 1.4525]$$

Since 0 is included in all computed intervals, the constraint is box-consistent in [4,5]×[2,3]×[9,10]

3.2. Is the system hull-consistent in box $B[x, y, v_0]$ = [4,5]×[2,3]×[9,10]?

hull-consistent in [4,5]×[2,3]×[9,10]

$$\Leftrightarrow \exists_{y\in[2,3]}\exists_{v_0\in[9,10]} y + 9.81\left(\frac{4}{v_0}\right)^2 - 4 = 0 \wedge \exists_{y\in[2,3]}\exists_{v_0\in[9,10]} y + 9.81\left(\frac{5}{v_0}\right)^2 - 5 = 0$$

$$\wedge \exists_{x\in[4,5]}\exists_{v_0\in[9,10]} 2 + 9.81\left(\frac{x}{v_0}\right)^2 - x = 0 \wedge \exists_{x\in[4,5]}\exists_{v_0\in[9,10]} 3 + 9.81\left(\frac{x}{v_0}\right)^2 - x = 0$$

$$\wedge \exists_{x\in[4,5]}\exists_{y\in[2,3]} y + 9.81\left(\frac{x}{9}\right)^2 - x = 0 \wedge \exists_{y\in[2,3]}\exists_{v_0\in[9,10]} y + 9.81\left(\frac{x}{10}\right)^2 - x = 0$$

However, $\exists_{x\in[4,5]}\exists_{v_0\in[9,10]} 3 + 9.81\left(\frac{x}{v_0}\right)^2 - x = 0$ cannot be satisfied:

$$3 + 9.81\left(\frac{x}{v_0}\right)^2 - x = 0 \Leftrightarrow \left(\frac{9.81}{{v_0}^2}\right)x^2 - x + 3 = 0$$

$$\Leftrightarrow x = \frac{1 \pm \sqrt{1 - 4 \times \left(\frac{9.81}{{v_0}^2}\right) \times 3}}{2\left(\frac{9.81}{{v_0}^2}\right)} = \frac{1 \pm \sqrt{1 - \frac{117.72}{{v_0}^2}}}{2\left(\frac{9.81}{{v_0}^2}\right)}$$

But if $v_0 \leq 10$ then $\frac{117.72}{{v_0}^2} > 1$ and $1 - \frac{117.72}{{v_0}^2} < 0$ and so there are no roots

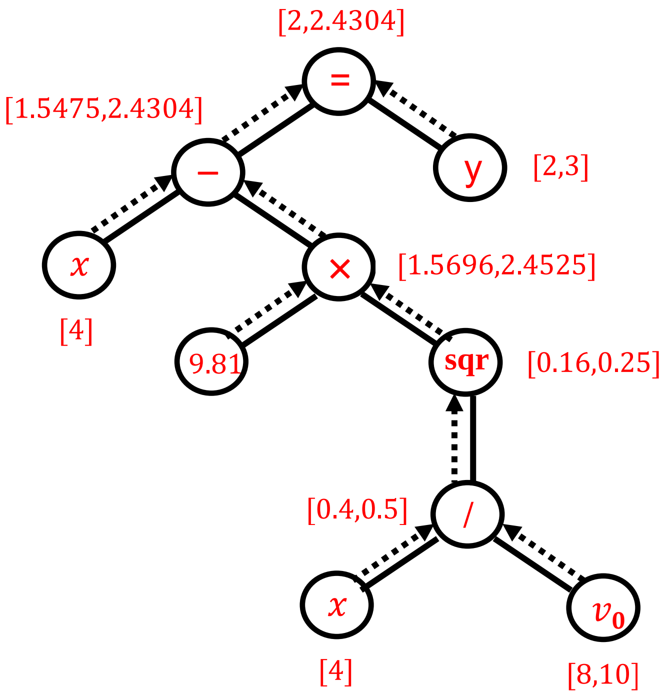
∴ the constraint is not hull-consistent in box B

3.3. Can you reduce box B by applying HC4-revise to the constraint? Justify.

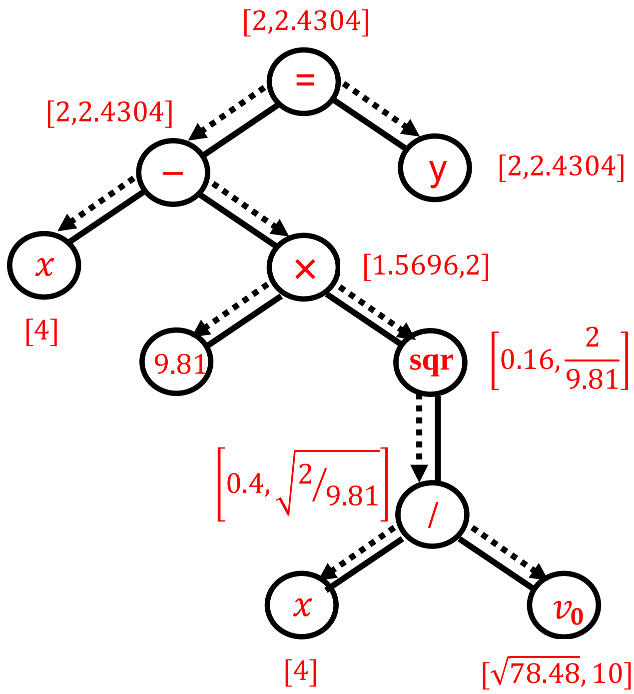
HC4-revise enforce hull-consistency on a constraint by implicitly decomposing it into primitive constraints. Since box-consistency is stronger than hull-consistency applied on the primitive constraints obtained by decomposition, and the constraint is box-consistent in box B, then B cannot be narrowed by the HC4-revise.

3.4. Apply HC4-revise to the constraint with an initial box $B'[x, y, v_0]$ = [4]×[2,3]×[8,10].

Forward evaluation:



Backward propagation:



The resulting box is [4]×[2,2.4304]×[$\sqrt{78.48}$, 10].

3.5. What is the box obtained by applying BC3-revise to the constraint with the same initial box B'?

BC3-revise enforces box-consistency on the original constraint and is stronger than HC4-revise. We have seen in the previous question that applying HC4-revise to the constraint results in the box [4]×[2,2.4304]×[$\sqrt{78.48}$, 10].

Let x=4 and $v_0=\sqrt{78.48}$: $x - 9.81\left(\frac{x}{v_0}\right)^2 = 4 - 9.81\left(\frac{4}{\sqrt{78.48}}\right)^2 = 4 - 9.81\frac{16}{78.48} = 4 - \frac{16}{8} = 4 - 2 = 2$

Thus x=4, y=2 and $v_0=\sqrt{78.48}$ is a solution of the constraint.

Let x=4 and v_0=10: $x - 9.81\left(\frac{x}{v_0}\right)^2 = 4 - 9.81\left(\frac{4}{10}\right)^2 = 4 - 9.81\frac{16}{100} = 4 - 1.5696 = 2.4304$

Thus x=4, y=2 and v_0=10 is also a solution of the constraint.

applying HC4-revise on the system cannot discard any of these solutions and the box cannot be narrowed thus the result will be the same: [4]×[2,2.4304]×[$\sqrt{78.48}$, 10].