Constraint Programming

2019/2020 – Mini-Test #2

Wednesday, 11 December, 16:30 h in 128-Ed.II

Duration: 1.5 h (open book)

1. Interval Arithmetic

Consider the univariate polynomial function expressed in the standard form as:

$$f(x) = x^2 - 4x + 3$$

1.1. Express the function in the factored form.

$$x = \frac{4 \mp \sqrt{16-4(3)}}{2} = \frac{4 \mp 2}{2} = 2 \mp 1 = 1 \vee 3$$

$\therefore$ $\quad f(x) = (x-1)(x-3)$ ← factored form

1.2. Compute the mean value extension of f over the interval $[a,b]$ centered at the midpoint c.

$$F_c(x) = f(c) + F'([a,b]) \times (x-c)$$

$$c = \frac{a+b}{2}$$

$$f(c) = c^2 - 4c + 3$$

$$F'([a,b]) = 2[a,b] - 4$$

$$F_c(x) = c^2 - 4c + 3 + (2[a,b] - 4) \times (x-c)$$

1.3. Find, if possible, an interval (with width = 1) for which the natural interval evaluation of the mean value extension computes an enclosure smaller than the obtained by the factored form.

$[a,b] = [1.5,2.5] \qquad c = 2$

$(x-1)(x-3) = ([1.5,2.5]-1)([1.5,2.5]-3) = ([0.5,1.5])([-1.5,-0.5])$

$= [-2.25,-0.25]$ (width 2)

$F_c([1.5,2.5]) = 2^2 - 4(2) + 3 + (2[1.5,2.5]-4) \times ([1.5,2.5]-2)$

$= 4 - 8 + 3 + ([3,5]-4) \times [-0.5,0.5]$

$= -1 + [-1,1] \times [-0.5,0.5]$

$= -1 + [-0.5,0.5] = [-1.5,-0.5]$ (width 1)

1.4. Prove the inclusion monotonicity property of the interval arithmetic square operator.

Prove: $[a,b] \subseteq [c,d] \Rightarrow [a,b]^2 \subseteq [c,d]^2$

$[a,b] \subseteq [c,d] \Rightarrow c \le a \le b \le d$

Case $0 \le c$

$[a,b]^2 = [a^2,b^2]$

$[c,d]^2 = [c^2,d^2]$

$0 \le c \le a \Rightarrow c^2 \le a^2$

$0 \le b \le d \Rightarrow b^2 \le d^2$

$\therefore [a,b]^2 \subseteq [c,d]^2$

Case $c < 0 \leq a$

$[a, b]^2 = [a^2, b^2]$

$[c, d]^2 = [0, \max(c^2, d^2)]$

$0 \leq a \Rightarrow 0 \leq a^2$

$0 \leq b \leq d \Rightarrow b^2 \leq d^2$

$\therefore [a, b]^2 \subseteq [c, d]^2$

Case $a < 0 \leq b$

$[a, b]^2 = [0, \max(a^2, b^2)]$

$[c, d]^2 = [0, \max(c^2, d^2)]$

$c \leq a < 0 \Rightarrow a^2 \leq c^2$

$0 \leq b \leq d \Rightarrow b^2 \leq d^2$

$\therefore [a, b]^2 \subseteq [c, d]^2$

Case $b < 0 \leq d$

$[a, b]^2 = [b^2, a^2]$

$[c, d]^2 = [0, \max(c^2, d^2)]$

$0 \leq b^2$

$c \leq a < 0 \Rightarrow a^2 \leq c^2$

$\therefore [a, b]^2 \subseteq [c, d]^2$

Case $d < 0$

$[a, b]^2 = [b^2, a^2]$

$[c, d]^2 = [d^2, c^2]$

$b \leq d < 0 \Rightarrow d^2 \leq b^2$

$c \leq a < 0 \Rightarrow a^2 \leq c^2$

$\therefore [a, b]^2 \subseteq [c, d]^2$

Thus, for all possible cases: $[a, b]^2 \subseteq [c, d]^2$

$\therefore [a, b] \subseteq [c, d] \Rightarrow [a, b]^2 \subseteq [c, d]^2$ q.e.d.

2. Interval Newton

Consider the function: $f(x) = (x - 1)^2 - e^{x-3}$

2.1. Define the interval Newton function for the polynomial.

$$N([a, b]) = c - \frac{f(c)}{F'([a, b])} \qquad with \quad c = \frac{a + b}{2}$$

$f(c) = (c - 1)^2 - e^{c-3}$

$F'([a, b]) = 2([a, b] - 1) - e^{[a,b]-3}$

$\therefore N([a, b]) = c - \frac{(c-1)^2 - e^{c-3}}{2([a,b]-1) - e^{[a,b]-3}}$

2.2. Evaluate the interval Newton function in [0.4, 0.8] and in [0.8,1.2].

$$N([0.4,0.8]) = 0.6 - \frac{(0.6-1)^2 - e^{0.6-3}}{2([0.4,0.8]-1) - e^{[0.4,0.8]-3}}$$
$$= 0.6 - \frac{0.0693}{[-1.2,-0.4] - [e^{-2.6}, e^{-2.2}]} = 0.6 - \frac{0.0693}{[-1.3108,-0.4743]}$$
$$= 0.6 - [-0.1461,-0.0529] = [0.6529,0.7461]$$

$$N([0.8,1.2]) = 1 - \frac{(1-1)^2 - e^{1-3}}{2([0.8,1.2]-1) - e^{[0.8,1.2]-3}}$$
$$= 1 - \frac{-e^{-2}}{[-0.4,0.4] - [e^{-2.2}, e^{-1.8}]} = 1 - \frac{-e^{-2}}{[-0.5653,0.2892]}$$
$$= 1 - ([-\infty,-0.4679] \cup [0.2394,+\infty]) = [-\infty, 0.7606] \cup [1.4679,+\infty]$$

2.3. From the above evaluations what can be concluded with respect to the existence of roots within those intervals.

$N([0.4,0.8]) = [0.6529,0.7461] \subset [0.4,0.8] \Rightarrow$ There is at least one root in [0.4,0.8]

$N([0.8,1.2]) \cap [0.8,1.2] = ([-\infty, 0.7606] \cup [1.4679,+\infty]) \cap [0.8,1.2] = \emptyset \Rightarrow$ There are no roots in [0.8,1.2]

3. Constraint Propagation

Consider the constraints below and a box B = [2,3]×[3,5]

c1: $(x-4)^2 - y \leq -1$

c2: $x^2 - 4x + y \leq 0$

3.1. Is the system hull-consistent in box B?

hull-consistent in [2,3]×[3,5]

$\Leftrightarrow \exists_{y\in[3,5]}(2-4)^2 - y \leq -1 \wedge \exists_{y\in[3,5]} 2^2 - 4(2) + y \leq 0$

$\wedge \exists_{y\in[3,5]}(3-4)^2 - y \leq -1 \wedge \exists_{y\in[3,5]} 3^2 - 4(3) + y \leq 0$

$\wedge \exists_{x\in[2,3]}(\mathrm{x}-4)^2 - 3 \leq -1 \wedge \exists_{x\in[2,3]} \mathrm{x}^2 - 4\mathrm{x} + 3 \leq 0$

$\wedge \exists_{x\in[2,3]}(x-4)^2 - 5 \leq -1 \wedge \exists_{x\in[2,3]} x^2 - 4x + 5 \leq 0$

Consider: $x^2 - 4x + 5$, it is nondecreasing in [2,3] since its derivative $2x - 4$ is non negative in [2,3] and positive in 3. Therefore:

$\forall_{x\in[2,3]} 2^2 - 4(2) + 5 \leq x^2 - 4x + 5 \leq 3^2 - 4(3) + 5$

$\forall_{x\in[2,3]} 1 \leq x^2 - 4x + 5 \leq 2$

$\therefore \nexists_{x\in[2,3]} x^2 - 4x + 5 \leq 0$

Thus the system is not hull-consistent in [2,3]×[3,5]

3.2. Is the system box-consistent in box B?

box-consistent in [2,3]×[3,5] $\Leftrightarrow$ the following inequalities are satisfiable:

$(2-4)^2 - [3,5] \leq -1 \wedge 2^2 - 4(2) + [3,5] \leq 0$

$\wedge (3-4)^2 - [3,5] \leq -1 \wedge 3^2 - 4(3) + [3,5] \leq 0$

$\wedge ([2,3]-4)^2 - 3 \leq -1 \wedge [2,3]^2 - 4[2,3] + 3 \leq 0$

$\wedge ([2,3]-4)^2 - 5 \leq -1 \wedge [2,3]^2 - 4[2,3] + 5 \leq 0$

$\Leftrightarrow 4 - [3,5] \leq -1 \wedge 4 - 8 + [3,5] \leq 0$
$\wedge\, 1 - [3,5] \leq -1 \wedge 9 - 12 + [3,5] \leq 0$
$\wedge\, [1,4] - 3 \leq -1 \wedge [4,9] - [8,12] + 3 \leq 0$
$\wedge\, [1,4] - 5 \leq -1 \wedge [4,9] - [8,12] + 5 \leq 0$

$\Leftrightarrow [-1,1] \leq -1 \wedge [-1,1] \leq 0$
$\wedge\, [-4,-2] \leq -1 \wedge [0,2] \leq 0]$
$\wedge\, [-2,1] \leq -1 \wedge [-5,4] \leq 0$
$\wedge\, [-4,-1] \leq -1 \wedge [-3,6] \leq 0$

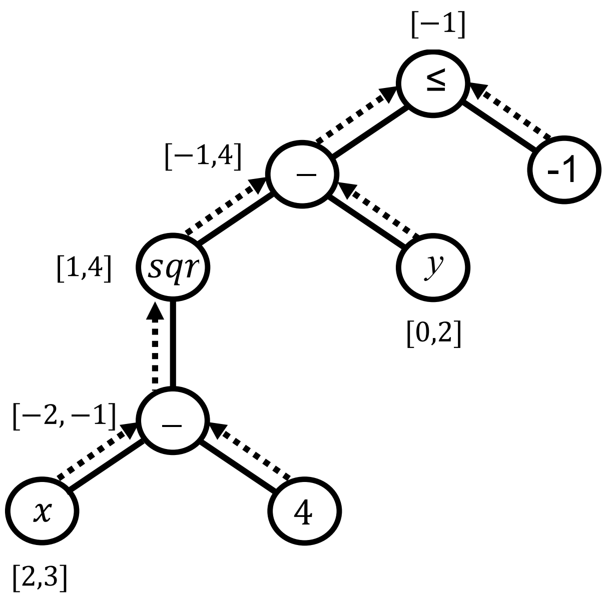
Since all inequalities are trivially satisfiable the system is box-consistent in [2,3]×[3,5]

3.3. Can you reduce box *B* by applying HC4-revise on both constraints? Justify.

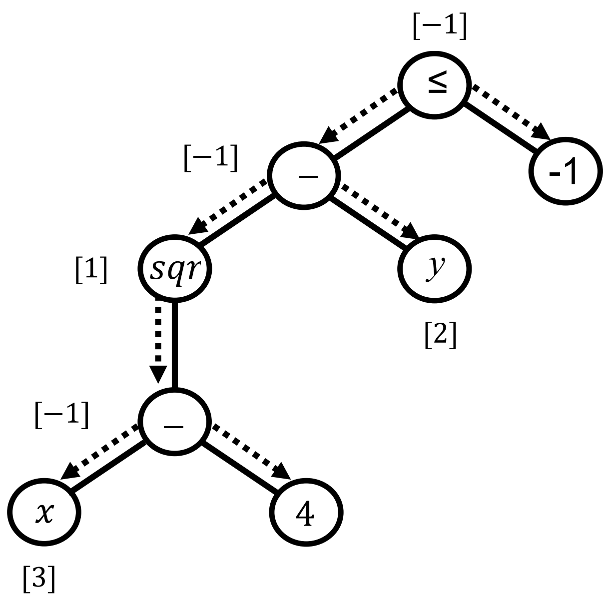
HC4-revise enforces hull-consistency in the decomposed system of primitive constraints which is weaker than box-consistency enforced on the original constraints. Since the system is already box-consistent in box *B*, the box cannot be reduced by applying the weaker HC4-revise.

3.4. Apply HC4-revise to constraint c1 with an initial box *B'* = [2,3]×[0,2].

Forward evaluation:



Backward propagation:



The resulting box is [3]×[2].

3.5. What is the box obtained by applying BC3-revise on both constraints with the initial box *B'*?

BC3-revise enforces box-consistency on the original system of constraints and is stronger than HC4-revise. We have seen in the previous question that applying HC4-revise to constraint c1 results in box [3]×[2]. Since *x*= 3, *y*=2 is a solution of the system:

c1: $(3-4)^2 - 2 \leq -1 \Leftrightarrow 1 - 2 \leq -1 \Leftrightarrow -1 \leq -1$ (True)

c2: $3^2 - 4(3) + 2 \leq 0 \Leftrightarrow 9 - 12 + 2 \leq 0 \Leftrightarrow -1 \leq 0$ (True)

applying HC4-revise on the system cannot discard it and the result is the box [3]×[2]. Therefore for any stronger method, such as BC3-revise, the result will be the same: [3]×[2].