Constraint Programming

2017/2018 – Mini-Test #2

Wednesday, 19 December, 18:00 h in 204-Ed.II

Duration: 1.5 h (open book)

1. Interval Arithmetic

Consider the univariate polynomial function expressed in the standard form as: $f(x) = x^3 - x^2 + x$

1.1. Express this function in the Horner form: $a + x(b + x(c + dx))$

$f(x) = x^3 - x^2 + x = x(x^2 - x + 1) = x(x(x-1) + 1)$

1.2. Choose an interval (with width > 0) for which the natural interval evaluation of the standard form computes an enclosure smaller than the obtained by the Horner form. Justify

$I = [-1,1]$

Stand. form: $[-1,1]^3 - [-1,1]^2 + [-1,1] = [-1,1] - [0,1] + [-1,1] = [-2,1] + [-1,1] = [-3,2]$

width: 5

Horner form: $[-1,1]([-1,1]([-1,1] - 1) + 1) = [-1,1]([-1,1][-2,0] + 1)$

$= [-1,1]([-2,2] + 1) = [-1,1][-1,3] = [-3,3]$

width: 6

1.3. Compute the smallest enclosure you can for the range of the function in $x \in [0,1]$. Justify.

$f'(x) = 3x^2 - 2x + 1$

$f'(x) = 0 \rightarrow x = \frac{2 \pm \sqrt{4-12}}{6}$ não tem raízes reais,

Logo $f(x)$ é monótona, e como $f'(0) = 1$ é crescente e portanto:

$f([0,1]) = [f(0), f(1)]$=[0,1]

2. Interval Newton

Consider the polynomial of the previous question: $f(x) = x^3 - x^2 + x$

2.1. Define the interval Newton function for the polynomial.

$$N(I) = c(I) - \frac{c(I)^3 - c(I)^2 + c(I)}{3I^2 - 2I + 1} \qquad c([a,b]) = \frac{a+b}{2}$$

2.2. Prove with the interval Newton method that the polynomial has no roots in [1/2, 3/4].

$$I = [0.5, 0.75] \qquad c(I) = \frac{0.5 + 0.75}{2} = 0.625$$

$$N([0.5,0.75]) = 0.625 - \frac{0.625^3 - 0.625^2 + 0.625}{3[0.5,0.75]^2 - 2[0.5,0.75] + 1}$$

$$N([0.5,0.75]) \cong 0.625 - \frac{0.4785}{3[0.25,0.5625] - [1,1.5] + 1}$$

$$N([0.5,0.75]) \cong 0.625 - \frac{0.4785}{[0.25,1.6875]} \cong 0.625 - [0.2835,1.9141] \cong [-1.2890,0.3414]$$

Como $N([0.5,0.75]) \cap [0.5,0.75] = \emptyset$ provámos que não há raízes em [0.5,0.75]

2.3. Prove with the interval Newton method that the polynomial has a root in [-1/2, 1/3].

$$I = [-0.5, 1/3] \qquad c(I) = \frac{\frac{1}{3} - 0.5}{2} = -\frac{1}{12}$$

$$N([-0.5,1/3]) = -\frac{1}{12} - \frac{\left(-\frac{1}{12}\right)^3 - \left(-\frac{1}{12}\right)^2 + \left(-\frac{1}{12}\right)}{3[-0.5,1/3]^2 - 2[-0.5,1/3] + 1}$$

$$N([-0.5,1/3]) \cong -0.833 - \frac{-0.091}{3[0,0.25] - [-0.5,0.3333] + 1}$$

$$N([-0.5,1/3]) \cong -0.833 - \frac{-0.091}{[0.3333,2.75]} \cong -0.833 - [-0.2725, -0.0331] \cong [-0.0502, 0.1893]$$

Como $N([-0.5,1/3]) \subset [-0.5,1/3]$ provámos que há pelo menos uma raíz em $[-0.5,1/3]$

3. Constraint Propagation

Consider the constraints below and a box $B = [-3,3]\times[-3,3]$

c1: $x^3 - 3x^2 + 3x - y \leq 1$

c2: $x^2 + y^2 \leq 9$

3.1. Is the system box-consistent in box B?

Para o sistema ser box-consistent é necessário que todas as restrições o sejam.

A restrição c1 é box-consistent em B sse:

$[-3]^3 - 3[-3]^2 + 3[-3] - [-3,3] \leq 1$ $\qquad$ $[3]^3 - 3[3]^2 + 3[3] - [-3,3] \leq 1$

$[-3,3]^3 - 3[-3,3]^2 + 3[-3,3] - [-3] \leq 1$ $\qquad$ $[-3,3]^3 - 3[-3,3]^2 + 3[-3,3] - [3] \leq 1$

No entanto: $[3]^3 - 3[3]^2 + 3[3] - [-3,3] = 27 - 27 + 9 - [-3,3] = 9 + [-3,3] = [6,12] > 1$

Logo c1 não é box-consistent e portanto o sistema não é box-consistent.

3.2. Is the system hull-consistent in box B?

Box-consistency é mais fraco que hull-consistency para o mesmo conjunto de restrições, logo se o sistema não é box-consistent então também não é hull-consistent.

3.3. Find narrowing functions for the constraints, by decomposing them into primitive constraints, and check whether the system is hull-consistent wrt these narrowing functions in B.

c1: $x^3 - 3x^2 + 3x - y \leq 1$

- c1.1: $x_1 - y \leq 1$
- c1.2: $x_1 = x \times x_2$
- c1.3: $x_2 = x_3 + 3$
- c1.4: $x_3 = x \times x_4$
- c1.5: $x_4 = x - 3$

c2: $x^2 + y^2 \leq 9$

- c2.1: $x_5 + x_6 \leq 9$
- c2.2: $x_5 = x \times x_7$
- c2.3: $x_6 = y \times x_8$
- c2.4: $x_7 = x$
- c2.5: $x_8 = y$

As narrowing functions são obtidas resolvendo cada restrição em ordem a cada variável.

Exemplo: $x_1 - y \leq 1$

$x_1 - y = [-\infty,1]$

$x_1 = [-\infty,1] + y$

$NF_{1.1:x1}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,Y,([-\infty,1]+Y)\cap X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>$

$x_1 - y \leq 1$

$x_1 - y = [-\infty,1]$

$y = x_1 - [-\infty,1]$

$NF_{1.1:y}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X, (X_1-[-\infty,1])\cap Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>$

Ficamos então com as seguintes narrowing functions:

$NF_{1.1:x1}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,Y,([-\infty,1]+Y)\cap X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>$

$NF_{1.1:y}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,(X_1-[-\infty,1])\cap Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>$

$NF_{1.2:x}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <(X_1/X_2)\cap X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>$

$NF_{1.2:x1}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,Y,(X\times X_2)\cap X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>$

$NF_{1.2:x2}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,Y,X_1,(X_1/X)\cap X_2,X_3,X_4,X_5,X_6,X_7,X_8>$

$NF_{1.3:x2}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,Y,X_1,(X_3+3)\cap X_2,X_3,X_4,X_5,X_6,X_7,X_8>$

$NF_{1.3:x3}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,Y,X_1, X_2,(X_2-3)\cap X_3,X_4,X_5,X_6,X_7,X_8>$

$NF_{1.4:x}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <(X_3/X_4)\cap X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>$

$NF_{1.4:x3}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,Y, X_1,X_2,(X\times X_4)\cap X_3,X_4,X_5,X_6,X_7,X_8>$

$NF_{1.4:x4}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,Y,X_1, X_2,X_3,(X_3/X)\cap X_4,X_5,X_6,X_7,X_8>$

$NF_{1.5:x}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <(X_4+3)\cap X,Y,X_1, X_2,X_3,X_4,X_5,X_6,X_7,X_8>$

$NF_{1.5:x4}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,Y,X_1, X_2,X_3,(X-3)\cap X_4,X_5,X_6,X_7,X_8>$

$NF_{2.1:x5}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,Y, X_1,X_2,X_3,X_4,([-\infty,9]-X_6)\cap X_5,X_6,X_7,X_8>$

$NF_{2.1:x6}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,Y,X_1,X_2,X_3,X_4,X_5,([-\infty,9]-X_5)\cap X_6,X_7,X_8>$

$NF_{2.2:x}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <(X_5/X_7)\cap X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>$

$NF_{2.2:x5}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,Y, X_1,X_2, X_3,X_4,(X\times X_7)\cap X_5,X_6,X_7,X_8>$

$NF_{2.2:x7}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,Y,X_1, X_2,X_3, X_4,X_5,X_6,(X_5/X)\cap X_7,X_8>$

$NF_{2.3:y}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,(X_6/X_8)\cap Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>$

$NF_{2.3:x6}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,Y, X_1,X_2, X_3,X_4, X_5,(Y\times X_8)\cap X_6,X_7,X_8>$

$NF_{2.3:x8}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,Y,X_1, X_2,X_3, X_4,X_5,X_6,X_7,(X_6/Y)\cap X_8>$

$NF_{2.4:x}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X_7\cap X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>$

$NF_{2.4:x7}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,Y,X_1, X_2,X_3,X_4,X_5,X_6,X\cap X_7,X_8>$

$NF_{2.5:y}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,X_8\cap Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>$

$NF_{2.5:x8}(<X,Y,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8>) = <X,Y,X_1, X_2,X_3,X_4,X_5,X_6,X_7,Y\cap X_8>$

O sistema é hull-consistent em B sse ao executarmos o algoritmo de propagação de restrições com a box

$B'=<[-3,3],[-3,3],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty]>$

obtivermos um ponto fixo em que os domínios das variáveis x e y não são alterados:

$B'=NF_{2.5:x8}(B')=<[-3,3],[-3,3],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-3,3]>$

$B'=NF_{2.5:x7}(B')=<[-3,3],[-3,3],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-3,3],[-3,3]>$

$B'=NF_{2.3:x6}(B')=<[-3,3],[-3,3],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-9,+9],[-3,3],[-3,3]>$

$B'=NF_{2.2:x5}(B')=<[-3,3],[-3,3],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-9,9],[-9,9],[-3,3],[-3,3]>$

$B'=NF_{1.5:x4}(B')=<[-3,3],[-3,3],[-\infty,+\infty],[-\infty,+\infty],[-\infty,+\infty],[-6,0],[-9,9],[-9,9],[-3,3],[-3,3]>$

$B'=NF_{1.4:x3}(B')=<[-3,3],[-3,3],[-\infty,+\infty],[-\infty,+\infty],[-18,18],[-6,0],[-9,9],[-9,9],[-3,3],[-3,3]>$

$B'=NF_{1.3:x2}(B')=<[-3,3],[-3,3],[-\infty,+\infty],[-15,21],[-18,18],[-6,0],[-9,9],[-9,9],[-3,3],[-3,3]>$

$B'=NF_{1.2:x1}(B')=<[-3,3],[-3,3],[-63,63],[-15,21],[-18,18],[-6,0],[-9,9],[-9,9],[-3,3],[-3,3]>$

$B'=NF_{1.1:x1}(B')=<[-3,3],[-3,3],[-63,4],[-15,21],[-18,18],[-6,0],[-9,9],[-9,9],[-3,3],[-3,3]>$

Chegámos a um ponto fixo logo o sistema é hull-consistent em B.